C. Ganesa Moorthy
G. Udhaya Sankar

Numerical Methods for Calculus Students

LAP LAMBERT Academic Publishing

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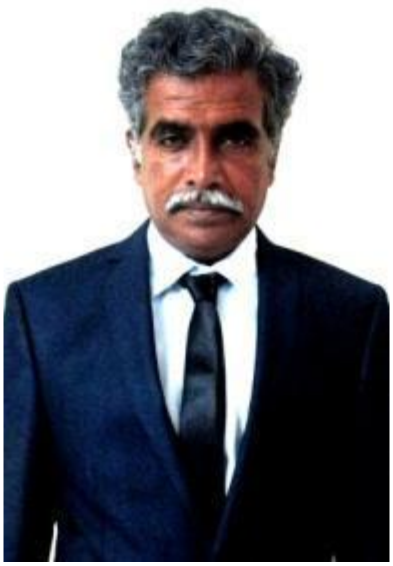

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Preface

This book is intended for those who completed school studies and who had calculus course and trigonometry course in schools. In the days when there were no computers, numerical methods were used to simplify computational works done by hand and mind. When main frame computers were introduced, there was a need to reduce usage of memory space and computer time in calculations, for which numerical methods were used. Even today, many recent researches claim that the algorithms described in those research works reduce computer time. There are many software packages to do computational works for almost all kinds of fundamental mathematical problems using numerical methods. Because of the existence of such packages, there are many natural questions. Do students who completed school level mathematics necessarily need a book in numerical methods? This question is raised because a student can solve numerical problems by using packages without knowing subject. Packages never give accuracy of the results obtained by users, and there are questions about reliability of the algorithms used in packages. If a student is very particular in reliability of the answers he/ she should write his/her own computer programs for his/her own chosen algorithm. If a student is particular in developing a software package containing programs to solve numerical problems, he/she should know the fundamental principles for numerical methods. This book is not an advanced level book, but it provides a sufficient good introduction of deriving numerical methods. It should be emphasized that there is no condition that only numerical methods described in books should be applied. It has been mentioned at the end of the fourth chapter of this book that some non formal methods may be used in deriving numerical differentiation methods. In the beginning of the fifth chapter, the fundamental principle in deriving numerical integration formulae has been explained. This will help a student to derive his/her own formulae for numerical integration. This does not mean that the book has got deviated from formal derivations. Almost all fundamental formal numerical methods to solve algebraic equations, to solve transcendental equations, to solve linear systems, to interpolate functions, to find numerical values for differentiation, to find numerical values for integration, and to solve ordinary differential equations with initial conditions, have been presented in this book. The final chapter includes Picard's method of successive approximations. Fundamental results for interpolating polynomials have been mentioned without proof. To avoid complexity, error analysis has not been included. But, error analysis related physical interpretations of order of numerical methods for solving differential equations have been explained in the sixth chapter. Additional steps have been inserted in worked-out examples to increase readability. This book also fulfills contents of syllabus of some courses for numerical methods prescribed in some universities. The present book has been written by **Dr. C. Ganesa Moorthy** (Professor, Department of Mathematics, Alagappa University, INDIA) with the financial support of RUSA –phase 2.0 grant sanctioned vide Letter No. F. 24-51/2014-U, Policy (TNMulti-Gen), Dept. of Edn, Govt. of India, Dt. 09.10.2018.

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CHAPTER- 1

Algebraic and Transcendental Equations

The expression $x^3 - 3x^2 + 2$ is a polynomial (function) and $x^3 - 3x^2 + 2 = 0$ is a polynomial equation. The value 1for x is a root or a zero of the polynomial $x^3 - 3x^2 + 2$ or the polynomial equation $x^3 - 3x^2 + 2 = 0$. This means that if we substitute $x = 1$ in the equation $x^3 - 3x^2 + 2 = 0$, then this equation is satisfied by this value. The function $\tan x - e^x + 1$ is not a polynomial, and hence it is called a transcendental function. The corresponding equation $\tan x - e^x + 1 = 0$ is a transcendental equation. The value 0 for x is a root or zero of the function $\tan x - e^x + 1$ or of the equation $\tan x - e^x + 1 = 0$. We also say that $x = 1$ is a solution of the polynomial equation $x^3 - 3x^2 = -2$, and $x = 0$ is a solution of the transcendental equation $\tan x - e^x + 1 = 0$.

This chapter is to present numerical methods to solve such equations. Numerical methods provide numerical solutions which are numbers approximately equal to solutions. In general, we consider a function $f(x)$ or an equation $f(x) = 0$. We shall describe the following four methods to find solutions or roots of the equation $f(x) = 0$.

(i) The bisection method,

(ii) The iteration method,

(iii) The method of false position, and

(iv) The Newton-Raphson method.

The bisection method:

Consider an equation $f(x) = 0$.

Initial step:

Find two points x_1 and x_2 by a trial and error method such that $f(x_1) \geq 0$ and $f(x_2) \leq 0$. If $f(x_1) = 0$ or $f(x_2) = 0$, then we stop our procedure. If $f(x_1) = 0$, then x_1 is a solution; and if $f(x_2) = 0$, then x_2 is a solution. Suppose $f(x_1) > 0$ and $f(x_2) < 0$. Then, we expect a solution in the interval with end points x_1 and x_2.

Step A:

Find $x_3 = \frac{x_1+x_2}{2}$, the point which bisects the interval with the end points x_1 and x_2. If $f(x_3) \approx 0$, then we stop our procedure and x_3 is an approximate value for the solution. Otherwise, we consider the following two cases.

Case (i):

Suppose $f(x_3) > 0$. Then x_2 remains as x_2, and give a new value x_3 for x_1, and repeat Step A.

Case (ii):

Suppose $f(x_3) < 0$. Then $x_1$remains as x_1, and give a new value x_3 for x_2, and repeat Step A.

Step A is repeated until we get a good value for x_3 such that $f(x_3) \approx 0$.

Remark:

This method works very slowly, but the convergence of this method is always assured for continuous functions.

Example:

Solve $x^3 - 9x + 1 = 0$ for a root between $x = 2$ and $x = 4$.

Solution:

Write $f(x) = x^3 - 9x + 1$. Then $f(2) = 2^3 - 9 \times 2 + 1 = -9 < 0$ and $f(4) = 4^3 - 9 \times 4 + 1 = 29 > 0$. So, take $x_1 = 4$ and $x_2 = 2$ in the first iteration. Then $x_3 = \frac{x_1+x_2}{2} = \frac{4+2}{2} = 3$. Now, $f(3) = 3^3 - 9 \times 3 + 1 = 1 > 0$. So, we now consider the new $x_1 = 3$ and $x_2 = 2$. Then $x_3 = \frac{x_1+x_2}{2} = \frac{3+2}{2} = 2.5$. Now, $f(2.5) = (2.5)^3 - 9 \times 2.5 + 1 = -5.875 < 0$. So, we now consider the new $x_1 = 3$ and $x_2 = 2.5$. Then $x_3 = \frac{x_1+x_2}{2} = \frac{3+2.5}{2} = 2.75$. Now, $f(2.75) = -2.9531 < 0$. So, we now consider the new $x_1 = 3$ and $x_2 = 2.75$. Then $x_3 = \frac{x_1+x_2}{2} = \frac{3+2.75}{2} = 2.875$. Now, $f(2.875) = -1.1113 < 0$. So, we now consider the new $x_1 = 3$ and $x_2 = 2.875$. Then $x_3 = \frac{x_1+x_2}{2} = \frac{3+2.875}{2} = 2.9375$. Now, $f(2.9375) = -0.0901 < 0$. So, we now consider the new $x_1 = 3$ and $x_2 = 2.9375$. Then $x_3 = \frac{x_1+x_2}{2} = \frac{3+2.9375}{2} = 2.96875$. Now, $f(2.96875) = 0.446258 > 0$. So, we now consider the new $x_1 = 2.96875$ and $x_2 = 2.9375$. Then $x_3 = \frac{x_1+x_2}{2} = \frac{2.96875+2.9375}{2} = 2.953125$. Now, $f(2.953125) = 0.175922 > 0$. So, we now consider the new $x_1 = 2.953125$ and $x_2 = 2.9375$. Then $x_3 = \frac{x_1+x_2}{2} = \frac{2.953125+2.9375}{2} = 2.9453125$. Now, $f(2.9453125) = -0.04237794876 \approx 0$. Therefore, we consider 2.9453125 as a good approximate solution to the given equation.

The iteration method:

This title means a fixed point iteration method for us. Consider an equation $f(x) = 0$. Rewrite this equation in the form $g(x) = x$. (For example, $\sin x - e^x = 0$ can be written as $x + \sin x - e^x = x$). A value x^* is called a fixed point of the function $g(x)$ if $g(x^*) = x^*$; that is, g fixes the value of x^* as x^*. Any fixed point of $g(x)$ becomes a solution of $f(x) = 0$. So, we have to find a fixed point of $g(x)$. Fix any point x_0, which is initially considered as an approximate solution of $f(x) = 0$, find $x_1 = g(x_0)$, $x_2 = g(x_1)$, $x_3 = g(x_2)$, …. . If x_n converges to x^* and g is continuous, then $g(x_n)$ converges to $g(x^*)$ so that $x_{n+1} = g(x_n)$

converges to $x^* = g(x^*)$. Thus, if n is very large, then, in general, x_n is a good approximate solution to $g(x) = x$ or $f(x) = 0$.

Remark:

Suppose $|g'(x)|<1$ in some interval I in which we search for a solution of $f(x) = 0$. Then the iteration procedure converges. If our selection of $g(x)$ does not satisfy this condition, then the convergence is not assured. For example, for $f(x) = \cos x - 2x + 3$ given in the next example, if we take $g(x) = \cos x - x + 3$, then this condition is not satisfied near $\frac{\pi}{2}$, and hence the corresponding method may not converge. So, we take $g(x) = \frac{1}{2}(\cos x + 3)$ so that $|g'(x)| = \left|\frac{\sin x}{2}\right| < 1$.

Example:

Use the method of iteration to find a root of the equation $2x = \cos x + 3$ correct to three decimal places.

Solution: Write the given equation in the form $x = \frac{1}{2}(\cos x + 3)$ with $g(x) = \frac{1}{2}(\cos x + 3)$. We start with $x_0 = \frac{\pi}{2}$ (radians). We have the following iterations.

$x_1 = \frac{1}{2}(\cos x_0 + 3) = \frac{1}{2}\left(\cos\frac{\pi}{2} + 3\right) = 1.5.$

$x_2 = \frac{1}{2}(\cos x_1 + 3) = \frac{1}{2}(\cos(1.5) + 3) = 1.535.$

$x_3 = \frac{1}{2}(\cos x_2 + 3) = \frac{1}{2}(\cos(1.535) + 3) = 1.518.$

$x_4 = \frac{1}{2}(\cos x_3 + 3) = \frac{1}{2}(\cos(1.518) + 3) = 1.526.$

$x_5 = \frac{1}{2}(\cos x_4 + 3) = \frac{1}{2}(\cos(1.526) + 3) = 1.522.$

$x_6 = \frac{1}{2}(\cos x_5 + 3) = \frac{1}{2}(\cos(1.522) + 3) = 1.524.$

We take $x = 1.524$ as a good approximate solution to the given equation.

Remark:

A calculator or a cosine table may be used to evaluate cosine values. The scientific calculator has to be kept in radian mode to evaluate the values for cosine function.

Example:

Find the positive square root of 2 by using an iteration method.

Solution: The required value is the positive solution of the equation $x^2 = 2$ or $x^2 - 2 = 0$. Write this equation as $(x-1)(x+1) - 1 = 0$ or $x = 1 + \frac{1}{1+x}$. If we take $g(x) = 1 + \frac{1}{1+x}$, then we have $g'(x) = -\frac{1}{(1+x)^2}$ and $|g'(1)|=|\frac{1}{(1+1)^2}|<1$, and hence we expect a convergence near the value 1. Take $g(x) = 1 + \frac{1}{1+x}$ and $x_0 = 1$. We have the following iterations.

$x_1 = 1 + \frac{1}{1+x_0} = 1 + \frac{1}{1+1} = 1.5.$

$x_2 = 1 + \frac{1}{1+x_1} = 1 + \frac{1}{1+1.5} = 1.4.$

$x_3 = 1 + \frac{1}{1+x_2} = 1 + \frac{1}{1+1.4} = 1.4167$ (Correct to 4 decimals).

$x_4 = 1 + \frac{1}{1+x_3} = 1 + \frac{1}{1+1.4167} = 1.4138.$

$x_5 = 1 + \frac{1}{1+x_4} = 1 + \frac{1}{1+1.4138} = 1.4143.$

$x_6 = 1 + \frac{1}{1+x_5} = 1 + \frac{1}{1+1.4143} = 1.4142.$

$x_7 = 1 + \frac{1}{1+x_6} = 1 + \frac{1}{1+1.4142} = 1.4142.$

Thus 1.4142 is an approximate value of $\sqrt{2}$, which is correct to 4 decimal places.

Remark:

All four methods of this chapter are iteration methods. However, the phrase "an iteration method" in this chapter refers to a fixed point iteration method. We shall find that the Newton-Raphson method is a fixed point iteration method.

The method of false position (or) Regula-falsi method:

Consider an equation $f(x) = 0$.

Initial Step:

Find two points x_1 and x_2, by a trial and error method, such that $f(x_1) \geq 0$ and $f(x_2) \leq 0$. If $f(x_1) = 0$ or $f(x_2) = 0$, the we stop our procedure. If $f(x_1) = 0$, then x_1 is a solution; and if $f(x_2) = 0$, then x_2 is a solution. Suppose $f(x_1) > 0$ and $f(x_2) < 0$. Then, we expect a solution in the interval with end points x_1 and x_2.

Step A:

Find $x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = x_1 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_1)$, the value of x at the point of intersection of the x-axis and the straight line joining the points

$(x_1, f(x_1))$ and $(x_2, f(x_2))$ in the xy-plane. If $f(x_3) \approx 0$, then we stop our procedure, and x_3 is an approximate value for the solution. Otherwise, we consider the following two cases.

Case (i):

Suppose $f(x_3) > 0$. Then x_2 remains as x_2, and give a new value x_3 for x_1, and repeat Step A.

Case (ii):

Suppose $f(x_3) < 0$. Then $x_1$remains as x_1, and give a new value x_3 for x_2, and repeat Step A.

Step A is repeated until we get a good value for x_3 such that $f(x_3) \approx 0$.

Remark:

Compare this method with the bisection method. We may find that the difference occurs only in the formulae for computation of x_3. However, the convergence of the regula-falsi method is faster than the convergence of the bisection method.

Example:

Find a real root of the polynomial $x^3 - 2x - 5$.

Solution: Let us apply the regula-falsi method. Write $f(x) = x^3 - 2x - 5$. Note that $f(2) = 2^3 - 2 \times 2 - 5 = -1 < 0$ and $f(3) = 3^3 - 2 \times 3 - 5 = 16 > 0$, and hence a root lies between 2 and 3. Take $x_1 = 3$ and $x_2 = 2$. Now, $x_3 = x_1 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_1) = 3 - \frac{2-3}{f(2)-f(3)} f(3) = 3 - \frac{1}{1+16} \times 16 = 2.059$. Now, $f(x_3) = f(2.059) = -0.386 < 0$. Therefore, we take $x_1 = 3$ and $x_2 = 2.059$. Now, $x_3 = x_1 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_1) = 3 - \frac{2.059-3}{f(2.059)-f(3)} f(3) = 3 - \frac{0.941}{0.386+16} \times 16 = 2.0812$. Now, $f(x_3) = f(2.0812) < 0$ (verify). Therefore, we take $x_1 = 3$ and $x_2 = 2.0812$. Now, $x_3 = x_1 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_1) = 3 - \frac{2.0812-3}{f(2.0812)-f(3)} f(3) = 2.0904$. Now, $f(x_3) = f(2.0904) < 0$ (verify). Therefore, we take $x_1 = 3$ and $x_2 = 2.0904$. Now, $x_3 = x_1 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_1) = 3 - \frac{2.0904-3}{f(2.0904)-f(3)} f(3) = 2.0934$. Now, $f(x_3) = f(2.0934) = -0.0126 < 0$. Therefore, we take $x_1 = 3$ and $x_2 = 2.0934$. Now, $x_3 = x_1 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_1) = 3 - \frac{2.0934-3}{f(2.0934)-f(3)} f(3) = 3 - \frac{0.9066}{0.0126+16} \times 16 = 2.0941$. Now, $f(x_3) = f(2.0941) = -0.0051 \approx 0$. Therefore, we consider 2.0941 as a good approximate root to the given polynomial.

Example:

Use regula-falsi method to find a real root of the equation $\log x - \cos x = 0$ accurate to four decimal places after three successive approximations.

Solution: Write $f(x) = \log x - \cos x$. Observe that $f(1) = \log 1 - \cos 1 = 0 - 0.5403 = -0.5403 < 0$ and $f(2) = \log 2 - \cos 2 = 0.69315 + 0.41615 =$

$1.1093 > 0$. Therefore, we take $x_1 = 2$ and $x_2 = 1$. Now, $x_3 = x_1 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_1) = 2 - \frac{1-2}{-0.5403-1.1093} \times 1.1093 = 1.3275$, the first approximation. Then $f(x_3) = f(1.3275) = \log 1.3275 - \cos 1.3275 = 0.2833 - 0.2409 = 0.0424 > 0$. Therefore, we take $x_1 = 1.3275$ and $x_2 = 1$. Now, $x_3 = x_1 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_1) = 1.3275 - \frac{1-1.3275}{-0.5403-0.0424} \times 0.0424 = 1.3037$, the second approximation. Then $f(x_3) = f(1.3037) = 1.24816 \times 10^{-3} > 0$. Therefore, we take $x_1 = 1.3037$ and $x_2 = 1$. Now, $x_3 = x_1 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_1) = 1.3037 - \frac{1-1.3037}{-0.5403-0.001248} \times 0.001248 = 1.3030$, the third approximation. Then $f(x_3) = f(1.3030) = 0.000062045 \approx 0$. Therefore, the required real root is 1.3030.

The Newton-Raphson Method:

Consider an equation $f(x) = 0$. Suppose $f(x_0) \approx 0$ and $f(x_0 + h) = 0$. Then, $0 = f(x_0 + h) \approx f(x_0) + hf'(x_0)$, by the Taylor's theorem. Hence $h \approx -\frac{f(x_0)}{f'(x_0)}$ and $x_0 + h \approx x_0 - \frac{f(x_0)}{f'(x_0)}$, where $f(x_0 + h) = 0$. Here, $x_0 - \frac{f(x_0)}{f'(x_0)}$ is the x-value of the point of intersection of the x-axis and the tangent of the curve $y = f(x)$ at $(x_0, f(x_0))$. Define

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$,

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$,

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$,

……………………….

……………………….

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

The last-equation formula is called the Newton-Raphson iteration formula. In general, x_{n+1} is a better approximate solution than x_n; and x_n converges to the exact solution of $f(x) = 0$, as $n \to \infty$. Thus, x_n is a good approximate solution of $f(x) = 0$, when n is sufficiently large.

Let us summarize the method:

Select some x_0 (sometimes it may be given), which is an initial approximate solution to $f(x) = 0$. Find $x_1, x_2, ...$ successively by using the formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$; for $= 0, 1, 2, ...$. Stop the procedure, when n is sufficiently large or when $f(x_n) \approx 0$; and conclude that x_n is a good approximate solution to $f(x) = 0$.

Example:

Find a real root of the equation $x^3 - x - 1 = 0$, using Newton-Raphson method correct to four decimal places.

Solution: Write $f(x) = x^3 - x - 1$ so that the given equation becomes $f(x) = 0$. Then $f'(x) = 3x^2 - 1$. Now the Newton-Raphson formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ becomes $x_{n+1} = x_n - \frac{x_n^3 - x_n - 1}{3x_n^2 - 1}$. Let us now take $x_0 = 2$. Then

$$x_1 = x_0 - \frac{x_0^3 - x_0 - 1}{3x_0^2 - 1} = 2 - \frac{2^3 - 2 - 1}{3 \times 2^2 - 1} = 1.54545,$$

$$x_2 = x_1 - \frac{x_1^3 - x_1 - 1}{3x_1^2 - 1} = 1.54545 - \frac{1.54545^3 - 1.54545 - 1}{3 \times 1.54545^2 - 1} = 1.35961,$$

$$x_3 = x_2 - \frac{x_2^3 - x_2 - 1}{3x_2^2 - 1} = 1.35961 - \frac{1.35961^3 - 1.35961 - 1}{3 \times 1.35961^2 - 1} = 1.32579,$$

$$x_4 = x_3 - \frac{x_3^3 - x_3 - 1}{3x_3^2 - 1} = 1.32579 - \frac{1.32579^3 - 1.32579 - 1}{3 \times 1.32579^2 - 1} = 1.32471,$$

$$x_5 = x_4 - \frac{x_4^3 - x_4 - 1}{3x_4^2 - 1} = 1.32471 - \frac{1.32471^3 - 1.32471 - 1}{3 \times 1.32471^2 - 1} = 1.324718.$$

Since $f(x_5) = f(1.324718) = 1.838 \times 10^{-7} \approx 0$, we stop the procedure and declare x_5 as a good approximate solution. Thus, 1.3247 is a good approximate solution to the equation.

Example:

Find a root of the equation $x \sin x + \cos x = 0$.

Solution: Here $f(x) = x \sin x + \cos x$ and $f'(x) = x \cos x$. Therefore the Newton-Raphson formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ becomes $x_{n+1} = x_n - \frac{x_n \sin x_n + \cos x_n}{x_n \cos x_n}$. Let us take $x_0 = \pi \approx 3.1416$. Then

$x_1 = x_0 - \frac{x_0 \sin x_0 + \cos x_0}{x_0 \cos x_0} = \pi - \frac{\pi \sin \pi + \cos \pi}{\pi \cos \pi} \approx 2.8233,$

$x_2 = x_1 - \frac{x_1 \sin x_1 + \cos x_1}{x_1 \cos x_1} = 2.8233 - \frac{2.8233 \sin 2.8233 + \cos 2.8233}{2.8233 \cos 2.8233} = 2.7986.$

Since $f(x_2) = f(2.7986) = 2.7986 \sin 2.7986 + \cos 2.7986 = -0.0006 \approx 0$, we conclude that x_2 is a good approximate solution to the given equation. Thus, 2.7986 is a good approximate root of the given equation.

Remarks:

(1) The Newton-Raphson method has a fastest convergence, when this is compared with the remaining three methods. This facility is obtained at the cost of evaluation of an additional function $f'(x)$ in each iteration.

(2) The equation $f(x) = 0$ can be written as $x = x - \frac{f(x)}{f'(x)}$, and the corresponding fixed point iteration formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ is the Newton-Raphson formula. Thus, Newton-Raphson method is also a fixed point iteration method.

(3) Consider two equations with two unknowns: $f(x, y) = 0, g(x, y) = 0$. The Newton iteration formula to solve this system is given by the following matrix formula:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} - \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}^{-1}_{at\ (x_n, y_n)} \begin{pmatrix} f(x_n, y_n) \\ g(x_n, y_n) \end{pmatrix},$$ where we should begin with an approximation (x_0, y_0) for (x, y) and apply the iteration formula again and again till $f(x_n, y_n) \approx (0, 0)$ and $g(x_n, y_n) \approx (0, 0)$.

(4) For linear systems, one has more simple iteration formulae. Consider the linear system: $a_1x + a_2y + a_3z = d_1$; $b_1x + b_2y + b_3z = d_2$; $c_1x + c_2y + c_3z = d_3$. The Jacobi's iteration formula to solve this system is the following system of iteration formulae:

$$x_{n+1} = \frac{1}{a_1}(d_1 - a_2y_n - a_3z_n) \quad ;$$

$$y_{n+1} = \frac{1}{b_2}(d_2 - b_1x_n - b_3z_n);$$

$$z_{n+1} = \frac{1}{c_3}(d_3 - c_1x_n - c_2y_n).$$

This method given in (4) provide only approximate numerical solutions, but with an advantage of requiring less processing memory when they are compared with the methods which provide exact solutions which are discussed in the next chapter.

Final Remarks:

Bisection method provides an idea of developing other methods according to nature of our problems. The methods discussed in this chapter are classical methods. One may design methods according to the nature of problems, for practical purposes.

Exercises:

(1) Obtain a root, correct to 3 decimal places, for each of the following equations by using the bisection method: (a) $x^3 - 3x - 5 = 0$; (b) $x^3 - 4x - 9 = 0$.

(2) Use the method of false position to compute the root of the equation $\cos x - xe^x = 0$.

(3) Find a real root of the equation $x^3 - 3x - 5 = 0$, by using the Newton-Raphson method, by performing three iterations.

(4) Find a real root of the equation $x = e^{-x}$ by using the Newton-Raphson method.

(5) Compute a root of the equation $e^x = x^2$ to accuracy of 10^{-5} using an iterative method.

(6) Use an iterative method to find a real root of the following equations: (a) $x \sin x = 1$; (b) $x = \frac{1}{(1+x)^2}$; (c) $e^{-x} = 10x$.

(7) Obtain the positive cube root of 12 by using the Newton-Raphson method.

(8) Find a real root of the equation $x^3 - x^2 - 1 = 0$ by using the regula-falsi method.

(9) Find a solution of the equation $\sin x = 1 - x$.

(10) Solve: $\sin x = \frac{x}{2}$.

CHAPTER- 2

Systems of Linear Equations

Consider a system of linear equations:

$$a_{11}x_1 + a_{12}x_2 + \cdots \ldots \ldots \ldots . + a_{1n}x_n = b_1;$$

$$a_{21}x_1 + a_{22}x_2 + \cdots \ldots \ldots \ldots . + a_{2n}x_n = b_2;$$

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$$a_{n1}x_1 + a_{n2}x_2 + \cdots \ldots \ldots \ldots . + a_{nn}x_n = b_n.$$

This system can also be written in the matrix form as

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

i.e. $AX = B.$

This system consists of n equations with n unknowns. Here $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ is the coefficient matrix, $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is the unknown variables-column vector and $B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ is the given constants-column vector.

There are two major branches in the methods to solve this system. One type consists of iterative methods which may be repeated any number of times, and they give only approximate solutions. We shall discuss the other type of direct methods which give exact solutions within finite number of stages or iterations. We shall discuss the following direct methods in this chapter:

(i) Matrix inversion method;

(ii) Cramer's rule;

(iii) Gauss elimination method;

(iv) Gauss-Jordan method;

(v) Triangularization method.

The first two methods are based on the determinants of the coefficient matrices. The second two methods are elimination methods. The last method is based on factorization, which is lengthy but it requires less processing memory.

There are three possibilities in the nature of the solutions of a linear system: (i) Infinitely many solutions; (ii) Unique solution; (iii) No solution. We shall discuss mainly the second type systems, which happens when and only when the determinant of the coefficient matrix is not equal to zero.

Matrix inversion method:

Consider the system

$$a_{11}x_1 + a_{12}x_2 + \cdots \ldots \ldots \ldots . + a_{1n}x_n = b_1;$$

$$a_{21}x_1 + a_{22}x_2 + \cdots \dots \dots \dots . + a_{2n}x_n = b_2;$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots \dots \dots \dots . + a_{nn}x_n = b_n$$

in the matrix form $AX = B$ with

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \ X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \text{ and } B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Suppose A^{-1} exists. Then the solution X of the system is given by $X = A^{-1}B$.

Remark:

One may use any method to find the A^{-1} of A. Finding inverses are equivalent to solving the corresponding systems. This fact may be observed by comparing the methods to find inverses with the methods to solve systems. We shall use the adjoint method to find inverse in the next example.

Example:

Solve the system

$x + y + z = 1;$

$4x + 3y - z = 2;$

$3x + 5y + 3z = -1;$

by matrix inversion method.

Solution: The given system is

$$\begin{pmatrix}1 & 1 & 1\\ 4 & 3 & -1\\ 3 & 5 & 3\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}1\\ 2\\ -1\end{pmatrix}. \text{ Write } A=\begin{pmatrix}1 & 1 & 1\\ 4 & 3 & -1\\ 3 & 5 & 3\end{pmatrix}.$$

Then $\det A = (9+5)-(12+3)+(20-9)=10$. Also,

$$adj\ A=\begin{pmatrix}(9+5) & -(3-5) & (-1-3)\\ -(12+3) & (3-3) & -(-1-4)\\ (20-9) & -(5-3) & (3-4)\end{pmatrix}=\begin{pmatrix}14 & 2 & -4\\ -15 & 0 & 5\\ 11 & -2 & -1\end{pmatrix}.$$

Therefore, $A^{-1}=\frac{1}{\det A}adj\ A=\frac{1}{10}\begin{pmatrix}14 & 2 & -4\\ -15 & 0 & 5\\ 11 & -2 & -1\end{pmatrix}$. Hence,

$$\begin{pmatrix}x\\ y\\ z\end{pmatrix}=A^{-1}\begin{pmatrix}1\\ 2\\ -1\end{pmatrix}=\frac{1}{10}\begin{pmatrix}14 & 2 & -4\\ -15 & 0 & 5\\ 11 & -2 & -1\end{pmatrix}\begin{pmatrix}1\\ 2\\ -1\end{pmatrix}=\frac{1}{10}\begin{pmatrix}22\\ -20\\ 8\end{pmatrix}=\begin{pmatrix}2.2\\ -2\\ 0.8\end{pmatrix}.$$

Thus, the solution of the system is $x=2.2, y=-2$ and $z=0.8$.

Example:

Solve the system

$3x+y+2z=3;$

$2x-3y-z=-3;$

$x+2y+z=4;$

by matrix inversion method.

Solution: The given system is

$$\begin{pmatrix}3 & 1 & 2\\ 2 & -3 & -1\\ 1 & 2 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}3\\ -3\\ 4\end{pmatrix}. \text{ Write } A=\begin{pmatrix}3 & 1 & 2\\ 2 & -3 & -1\\ 1 & 2 & 1\end{pmatrix}.$$

Then $\det A = 3 \times (-3+2) - (2+1) + 2 \times (4+3) = 8$. Also,

$$adj\ A = \begin{pmatrix} (-3+2) & -(1-4) & (-1+6) \\ -(2+1) & (3-2) & -(-3-4) \\ (4+3) & -(6-1) & (-9-2) \end{pmatrix} = \begin{pmatrix} -1 & 3 & 5 \\ -3 & -1 & 7 \\ 7 & -5 & -11 \end{pmatrix}.$$

Therefore, $A^{-1} = \frac{1}{\det A} adj\ A = \frac{1}{8}\begin{pmatrix} -1 & 3 & 5 \\ -3 & -1 & 7 \\ 7 & -5 & -11 \end{pmatrix}$. Hence,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} 3 \\ -3 \\ 4 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} -1 & 3 & 5 \\ -3 & -1 & 7 \\ 7 & -5 & -11 \end{pmatrix}\begin{pmatrix} 3 \\ -3 \\ 4 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 8 \\ 16 \\ -8 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$

Thus, the solution of the system is $x = 1, y = 2$ and $z = -1$.

Remark:

The adjoint method applied in the previous examples lead to the following Cramer's rule, which gives explicit calculations for the solutions. The derivation of this rule can be observed, but this derivation is omitted.

Cramer's rule:

Consider the system

$$a_{11}x_1 + a_{12}x_2 + \cdots \ldots \ldots \ldots + a_{1n}x_n = b_1;$$

$$a_{21}x_1 + a_{22}x_2 + \cdots \ldots \ldots \ldots + a_{2n}x_n = b_2;$$

…………………………………………………………………

…………………………………………………………………

$$a_{n1}x_1 + a_{n2}x_2 + \cdots \ldots \ldots \ldots + a_{nn}x_n = b_n$$

in the matrix form $AX = B$ with

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \text{ and } B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Let $A^{(j)}$ denote the matrix obtained from A on replacing the j-th column of A by the column vector B, for each $j = 1, 2, \dots, n$. Suppose A^{-1} exists. Then the solution X of the system is given by $x_j = \frac{\det A^{(j)}}{\det A}$, for $j = 1, 2, \dots, n$.

Example:

Solve the system

$x + y + z = 1;$

$4x + 3y - z = 2;$

$3x + 5y + 3z = -1;$

by Cramer's rule.

Solution: The given system is

$\begin{pmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$. Write $\mathrm{A} = \begin{pmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{pmatrix}$. Then $A^{(1)} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ -1 & 5 & 3 \end{pmatrix}$, $A^{(2)} = \begin{pmatrix} 1 & 1 & 1 \\ 4 & 2 & -1 \\ 3 & -1 & 3 \end{pmatrix}$ and $A^{(3)} = \begin{pmatrix} 1 & 1 & 1 \\ 4 & 3 & 2 \\ 3 & 5 & -1 \end{pmatrix}$.

Then, $\det A = 10$, $\det A^{(1)} = 22$, $\det A^{(2)} = -20$ and $\det A^{(3)} = 8$. Therefore,

$$x = \frac{\det A^{(1)}}{\det A} = \frac{22}{10} = 2.2,\ y = \frac{\det A^{(2)}}{\det A} = \frac{-20}{10} = -2,\ z = \frac{\det A^{(3)}}{\det A} = \frac{8}{10} = 0.8.$$

Thus, the solution of the system is $x = 2.2$, $y = -2$ and $z = 0.8$.

Example:

Solve the system

$3x + y + 2z = 3;$

$2x - 3y - z = -3;$

$x + 2y + z = 4;$

by Cramer's rule.

Solution: The given system is

$\begin{pmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 4 \end{pmatrix}$. Write $A = \begin{pmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{pmatrix}$. Then $A^{(1)} = \begin{pmatrix} 3 & 1 & 2 \\ -3 & -3 & -1 \\ 4 & 2 & 1 \end{pmatrix}$, $A^{(2)} = \begin{pmatrix} 3 & 3 & 2 \\ 2 & -3 & -1 \\ 1 & 4 & 1 \end{pmatrix}$ and $A^{(3)} = \begin{pmatrix} 3 & 1 & 3 \\ 2 & -3 & -3 \\ 1 & 2 & 4 \end{pmatrix}$.

Then, $\det A = 8$, $\det A^{(1)} = 8$, $\det A^{(2)} = 16$ and $\det A^{(3)} = -8$. Therefore,

$x = \frac{\det A^{(1)}}{\det A} = \frac{8}{8} = 1$, $y = \frac{\det A^{(2)}}{\det A} = \frac{16}{8} = 2$, $z = \frac{\det A^{(3)}}{\det A} = \frac{-8}{8} = -1$.

Thus, the solution of the system is $x = 1$, $y = 2$ and $z = -1$.

One can solve a system by his/her own elimination procedure. But, a computer needs a specific procedure to solve a system by means of elimination. We shall describe two elimination methods to write computer programs. One is Gauss elimination method. Another one is Gauss-Jordan elimination method or Jordan modification of Gauss elimination method.

Gauss elimination method:

Consider a system

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots \ldots \ldots \ldots . + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots \ldots \ldots \ldots . + a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots \ldots \ldots \ldots . + a_{nn}x_n = b_n \end{array}\right| \quad \text{(I)}$$

Step A:

Suppose $a_{11} \neq 0$. Otherwise, permute the equations so that the first coefficient a_{11} in the first equation is not equal to zero. Divide the first equation by a_{11} in (I). Multiply the new equation by $a_{21}, a_{31}, \ldots ., a_{n1}$, and subtract them respectively from 2-nd, 3-rd, …, n-th equations. We get a new system in the following form:

$$\left.\begin{array}{r} x_1 + a_{12}^{(1)}x_2 + a_{13}^{(1)}x_3 + \cdots \ldots \ldots \ldots . + a_{1n}^{(1)}x_n = b_1^{(1)} \\ a_{22}^{(1)}x_2 + a_{23}^{(1)}x_3 + \cdots \ldots \ldots \ldots . + a_{2n}^{(1)}x_n = b_2^{(1)} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right| \quad \text{(II)}$$

$$a_{n2}^{(1)}x_2 + a_{n3}^{(1)}x_3 + \cdots \ldots \ldots \ldots . + a_{nn}^{(1)}x_n = b_n^{(1)}$$

Thus, x_1 is eliminated from 2-nd, 3-rd, …,n-th equations and the coefficient of x_1 in the first equation is made into 1.

Then, we consider the last (n-1) equations in (II), apply Step A for these (n-1) equations with (n-1) unknowns, make the coefficient of x_2 in the second equation into 1, and eliminate x_2 from 3-rd, 4-th, …, n-th equations. The first equation is kept unchanged.

Then, we consider the last (n-2) equations in (II), apply Step A for these (n-2) equations with (n-2) unknowns, make the coefficient of x_3 in the second equation into 1, and eliminate x_3 from 4-th, 5-th, …, n-th equations. The first two equations are kept unchanged.

If we proceed in this way, we finally get a system in the following form.

$$\left.\begin{aligned} x_1 + c_{12}x_2 + c_{13}x_3 + \cdots \ldots \ldots . + c_{1n}x_n &= d_1 \\ x_2 + c_{23}x_3 + \cdots \ldots \ldots . + c_{2n}x_n &= d_2 \\ x_3 + \cdots \ldots \ldots . + c_{3n}x_n &= d_3 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ \cdots\cdots\cdots & \end{aligned}\right| \quad \text{(III)}$$

$$x_n = d_n$$

The solution can be obtained from (III) by back substitution. That is, substitute $x_n = d_n$ in the (n-1)-st equation of (III) and find x_{n-1} , substitute the values of x_n and x_{n-1} in the (n-2)-nd equation of (III) and find x_{n-2} , and proceed in this way until we obtain x_1 from the first equation of (III).

Remark:

This method works if and only if the determinant of the coefficient matrix is not equal to zero.

Example:

Solve the following system of equations using Gauss elimination method.

$$2x + 3y - z = 5$$

$$4x + 4y - 3z = 3$$

$$-2x + 3y - z = 1$$

Solution: In the first step, we divide the first equation by 2, multiply the resulting equation by 4 and -2, and subtract them respectively from 2nd and 3rd equations. We get:

$$x + \frac{3}{2}y - \frac{1}{2}z = \frac{5}{2}$$

$$-2y - z = -7$$

$$6y - 2z = 6$$

In the second step, we divide the second equation by -2, multiply the resulting equation by 6 and subtract it from the 3rd equation. We get:

$$x + \frac{3}{2}y - \frac{1}{2}z = \frac{5}{2}$$

$$y + \frac{1}{2}z = \frac{7}{2}$$

$$-5z = -15$$

In the third step we divide the third equation by -5. We get:

$$x + \frac{3}{2}y - \frac{1}{2}z = \frac{5}{2}$$

$$y + \frac{1}{2}z = \frac{7}{2}$$

$$z = 3$$

Back Substitution:

$z = 3,$

$y = \frac{7}{2} - \frac{1}{2}z = \frac{7}{2} - \frac{3}{2} = 2,$

$x = \frac{5}{2} + \frac{1}{2}z - \frac{3}{2}y = \frac{5}{2} + \frac{3}{2} - 3 = 1.$

Thus, the solution of the given system is $x = 1$, $y = 2$ and $z = 3$.

Example:

Solve the system of equations

$$x + y + z = 7$$

$$3x + 3y + 4z = 24$$

$$2x + y + 3z = 16$$

by Gauss elimination method.

Solution: In the first step, we multiply the first equation by 3 and 2, and subtract them respectively from 2nd and 3rd equations. We get:

$$x + y + z = 7$$

$$z = 3$$

$$-y + z = 2$$

In the second step interchange the 2nd and 3rd equations and divide the new 2nd equation by -1. We get:

$$x + y + z = 7$$

$$y - z = -2$$

$$z = 3$$

There is no need for third step and we go for back substitution:

$z = 3$,

$y = -2 + z = -2 + 3 = 1$,

$x = 7 - z - y = 7 - 3 - 1 = 3$.

Thus, the solution of the given system is $x = 3$, $y = 1$ and $z = 3$.

Gauss-Jordan elimination method (or) Jordan modification of Gauss elimination method:

Consider a system

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots \ldots \ldots \ldots . + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots \ldots \ldots \ldots . + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots \ldots \ldots \ldots . + a_{nn}x_n = b_n \end{array}\right| \quad \text{(I)}$$

Suppose $a_{11} \neq 0$. Otherwise, permute the equations so that the first coefficient a_{11} in the first equation is not equal to zero. Divide the first equation by a_{11} in (I). Multiply the new equation by $a_{21}, a_{31}, \ldots ., a_{n1}$, and subtract respectively them from 2-nd, 3-rd, …, n-th equations. We get a new system in the following form.

$$x_1 + a_{12}^{(1)}x_2 + a_{13}^{(1)}x_3 + \cdots \ldots \ldots \ldots . + a_{1n}^{(1)}x_n = b_1^{(1)}$$

$$a_{22}^{(1)}x_2 + a_{23}^{(1)}x_3 + \cdots \ldots \ldots \ldots . + a_{2n}^{(1)}x_n = b_2^{(1)}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{n2}^{(1)}x_2 + a_{n3}^{(1)}x_3 + \cdots \dots \dots \dots . + a_{nn}^{(1)}x_n = b_n^{(1)}.$$

Thus, x_1 is eliminated from 2-nd, 3-rd, …,n-th equations and the coefficient of x_1 in the first equation is made into 1.

Then, the coefficient of x_2 in the 2-nd equation is made into 1, by division. If it is not possible, permute the last (n-1) equations to make it possible without disturbing the first equation. Then eliminate x_2 from 1-st, 3-rd, 4-th, ….., n-th equations, by using the 2-nd equation with coefficient 1 for x_2 , by means of suitable multiplications of 2-nd equation and subtractions from the remaining equations.

In the third step, the coefficient of x_3 in the 3-rd equation is made into 1. If it is not possible, permute the last (n-2) equations to make it possible without disturbing the first two equations. Then eliminate x_3 from 1-st, 2-nd, 4-th, 5-th, …,n-th equations, by using the 3-rd equation.

If we proceed in this way, we finally get the solution in the form:

$$x_1 \qquad = d_1$$

$$x_2 \quad = d_2$$

$$\dots \dots \dots$$

$$\dots \dots \dots .$$

$$x_n = d_n$$

Remark:

This method works if only if the determinant of the coefficient matrix is not equal to zero.

Example:

Solve the following system of equations using Gauss-Jordan elimination method.

$$2x + 3y - z = 5$$

$$4x + 4y - 3z = 3$$

$$-2x + 3y - z = 1$$

Solution: In the first step, we divide the first equation by 2, multiply the resulting equation by 4 and -2, and subtract them respectively from 2nd and 3rd equations. We get:

$$x + \frac{3}{2}y - \frac{1}{2}z = \frac{5}{2}$$

$$-2y - z = -7$$

$$6y - 2z = 6$$

In the second step, we divide the second equation by -2, multiply the resulting equation by $\frac{3}{2}$ and 6 and subtract them respectively from the first equation and the third equation. We get:

$$x \quad - \frac{5}{4}z = -\frac{11}{4}$$

$$y + \frac{1}{2}z = \frac{7}{2}$$

$$-5z = -15$$

In the third step, we divide the 3-rd equation by -5, and multiply the resulting equation by $-\frac{5}{4}$ and $\frac{1}{2}$, and subtract them respectively from the first equation and the second equation. We get:

$$x \qquad = 1$$

$$y \quad = 2$$

$$z = 3$$

Thus, the solution of the given system is $x = 1, y = 2$ and $z = 3$.

Example:

Solve the system of equations

$$x + y + z = 7$$

$$3x + 3y + 4z = 24$$

$$2x + y + 3z = 16$$

by Gauss-Jordan elimination method.

Solution: In the first step, we multiply the first equation by 3 and 2, and subtract them respectively from 2nd and 3rd equations. We get:

$$x + y + z = 7$$

$$z = 3$$

$$-y + z = 2$$

In the second step interchange the 2nd and 3rd equations and divide the new 2nd equation by -1. Subtract the resulting new second equation from the first equation. We get:

$$x \quad + 2z = 9$$

$$y - z = -2$$

$$z = 3$$

In the third step multiply the 3-rd equation by 2 and -1 and subtract them respectively from the first equation and the second equation. We get:

$$x \qquad = 3$$

$$y \quad = 1$$

$$z = 3$$

Thus, the solution of the given system is $x = 3, y = 1$ and $z = 3$.

Triangularization method:

Consider the system

$$a_{11}x_1 + a_{12}x_2 + \cdots \ldots \ldots \ldots . + a_{1n}x_n = b_1;$$

$$a_{21}x_1 + a_{22}x_2 + \cdots \ldots \ldots \ldots . + a_{2n}x_n = b_2;$$

....

....

$$a_{n1}x_1 + a_{n2}x_2 + \cdots \dots \dots \dots . + a_{nn}x_n = b_n$$

in the matrix form $AX = B$ with

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix},\ X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \text{ and } B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Suppose that $\det A \neq 0$. Find a lower triangular matrix

$$L = \begin{pmatrix} l_{11} & 0 & 0 & & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & & 0 \\ & \vdots & & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{pmatrix}$$

and an upper triangular matrix

$$U = \begin{pmatrix} 1 & u_{12} & u_{13} & & u_{1n} \\ 0 & 1 & u_{23} & \cdots & u_{2n} \\ 0 & 0 & 1 & & u_{3n} \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

with diagonal elements equal to 1, such that $A = LU$. This can be achieved by comparing each entry in A and the corresponding entry in the product LU, and by solving the resulting equations. Write $LZ = B$ with $Z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$. Solve this system by forward substitution for Z. Then solve the system $UX = Z$ by backward substitution for X. This gives the solution X of the system $AX = B$, because $B = AX = (LU)X = L(UX) = LZ$.

Remark:

This method is also called as:

(i) Crout's reduction method;

(ii) Cholesky reduction method;

(iii) Factorization method;

(iv) Triangular factorization method.

This method works if and only if $\det A \neq 0$. Sometimes it may be assumed that all diagonal elements of L are equal to 1 and the diagonal elements of U are in the general form $u_{11}, u_{22}, \dots, u_{nn}$. The condition that all diagonal elements are equal to 1 should be imposed either for elements in U or for elements in L, not for both, and at least for one of them.

Example:

Solve the following system

$$5x_1 - 2x_2 + x_3 = 4$$

$$7x_1 + x_2 - 5x_3 = 8$$

$$3x_1 + 7x_2 + 4x_3 = 10$$

by triangularization method.

Solution: The given system in the matrix form is

$\begin{pmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 10 \end{pmatrix}$. Write $A = \begin{pmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{pmatrix}$, $L = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}$, and $U = \begin{pmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix}$.

Suppose $LU = A$. Comparing the entries in LU and in A, we have the following relations. $l_{11} = 5$ (Multiply the first row of L with the first column of U). $l_{11}u_{12} = -2$ and hence $5u_{12} = -2$ and $u_{12} = -\frac{2}{5}$ (Multiply first row of L with the second column of U). $l_{11}u_{13} = 1$, $5u_{13} = 1$, $u_{13} = \frac{1}{5}$ (Multiply the second row of L with the first column of U). $l_{21} = 7$ (Multiply the second row of L with the first column of U). $l_{21}u_{21} + l_{22} = 1, 7 \times \left(-\frac{2}{5}\right) + l_{22} = 1, l_{22} = \frac{19}{5}$ (Multiply the second row of L with the second column of U). $l_{21}u_{13} + l_{22}u_{23} = -5$, $7 \times \left(\frac{1}{5}\right) + \frac{19}{5} \times u_{23} = -5$, $u_{23} = -\frac{32}{19}$ (Multiply the second row of L with the third column of U). $l_{31} = 3$ (Multiply the third row of L with the first column of U). $l_{31}u_{12} + l_{32} = 7, 3 \times \left(-\frac{2}{5}\right) + l_{32} = 7, l_{32} = \frac{41}{5}$ (Multiply the third row of L with the second column of U). $l_{31}u_{13} + l_{32}u_{23} + l_{33} = 4$, $3 \times \left(\frac{1}{5}\right) + \left(\frac{41}{5}\right) \times \left(-\frac{32}{9}\right) + l_{33} = 4, l_{33} = \frac{327}{19}$ (Multiply the third row of L with the third column of U). Thus

$L = \begin{pmatrix} 5 & 0 & 0 \\ 7 & \frac{19}{5} & 0 \\ 3 & \frac{41}{5} & \frac{327}{19} \end{pmatrix}$ and $U = \begin{pmatrix} 1 & -\frac{2}{5} & \frac{1}{5} \\ 0 & 1 & \frac{32}{19} \\ 0 & 0 & 1 \end{pmatrix}$. Write $\begin{pmatrix} 5 & 0 & 0 \\ 7 & \frac{19}{5} & 0 \\ 3 & \frac{41}{5} & \frac{327}{19} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 10 \end{pmatrix}$. That is,

$5z_1 = 4,$

$7z_1 + \frac{19}{5}z_2 = 8,$

$3z_1 + \frac{41}{5}z_2 + \frac{327}{19}z_3 = 10.$

Then $z_1 = \frac{4}{5}$, $z_2 = (8 - 7z_1) \times \frac{5}{19} = \left(8 - 7 \times \frac{4}{5}\right) \times \frac{5}{19} = \frac{12}{19}$, and $z_3 = \left(10 - 3z_1 - \frac{41}{5}z_2\right) \times \frac{19}{327} = \left(10 - 3 \times \frac{4}{5} - \frac{41}{5} \times \frac{12}{19}\right) \times \frac{19}{327} = \frac{46}{327}$. Then, we should have $AX = (LU)X = L(UX) = LZ = B$, and $UX = Z$ (or)

$\begin{pmatrix} 1 & -\frac{2}{5} & \frac{1}{5} \\ 0 & 1 & \frac{32}{19} \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} \\ \frac{12}{19} \\ \frac{46}{327} \end{pmatrix}$. That is,

$x_1 - \frac{2}{5}x_2 + \frac{1}{5}x_3 = \frac{4}{5},$

$x_2 + \frac{32}{9}x_3 = \frac{12}{19},$

$x_3 = \frac{46}{327}.$

Therefore, $x_3 = \frac{46}{327}$, $x_2 = \frac{12}{19} - \frac{32}{19}x_3 = \frac{12}{19} - \frac{32}{19} \times \frac{46}{327} = \frac{284}{327}$, and $x_1 = \frac{4}{5} + \frac{2}{5}x_2 - \frac{1}{5}x_3 = \frac{4}{5} + \frac{2}{5} \times \frac{284}{327} - \frac{1}{5} \times \frac{46}{327} = \frac{366}{327}$. Thus, we have the solution $x_1 = \frac{366}{327}$, $x_2 = \frac{284}{327}$, and $x_3 = \frac{46}{327}$.

Example:

Solve the equations

$$2x + 3y + z = 9$$

$$x + 2y + 3z = 6$$

$$3x + y + 2z = 8$$

by triangularization method.

Solution: The given system is

$\begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \\ 8 \end{pmatrix}$. Write $\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$. Then we should have $u_{11} = 2, u_{12} = 3, u_{13} = 1, l_{21}u_{11} = 1$ so that $l_{21} = \frac{1}{2}$ (verify), $l_{21}u_{12} + u_{22} = 2$ so that $u_{22} = \frac{1}{2}$, $l_{21}u_{13} + u_{23} = 3$ so that $u_{23} = \frac{5}{2}$, $l_{31}u_{11} = 3$ so that $l_{31} = \frac{3}{2}$, $l_{31}u_{12} + l_{32}u_{22} = 1$ so that $l_{32} = -7$, $l_{31}u_{13} + l_{32}u_{23} + u_{33} = 2$ so that $u_{33} = 18$. Therefore, we have

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \\ 8 \end{pmatrix}.$$ Write $$\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \\ 8 \end{pmatrix}.$$

Then we have

$z_1 = 9,$

$\frac{1}{2}z_1 + z_2 = 6,$

$\frac{3}{2}z_1 - 7z_2 + z_3 = 8;$

or $z_1 = 9$, $z_2 = 6 - \frac{1}{2}z_1 = \frac{3}{2}$, $z_3 = 8 - \frac{3}{2}z_1 + 7z_2 = 5$. Thus, the given system reduces to:

$$\begin{pmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 9 \\ \frac{3}{2} \\ 5 \end{pmatrix}.$$ That is,

$2x + 3y + z = 9,$

$\frac{1}{2}y + \frac{5}{2}z = \frac{3}{2},$

$18z = 5.$

Therefore we have $z = \frac{5}{18}$, $y = 3 - 5z = \frac{29}{18}$, $x = (9 - 3y - z) \times \frac{1}{3} = \frac{35}{18}$.

Thus, we have the solution:

$x = \frac{35}{18}$, $y = \frac{29}{18}$, $z = \frac{5}{18}$.

Final Remarks:

All methods discussed in this chapter provide exact solutions of linear systems. These methods provide exact solutions at the cost of high processing memory. The Jacobi iteration method mentioned at the end of the previous chapter provide approximate numerical solutions, but with an advantage of less processing memory. Selection of methods is ours choice depending on our needs.

Exercises:

(1) Compute the inverse of the matrix

$$\begin{pmatrix} 3 & 2 & 4 \\ 2 & 1 & 1 \\ 1 & 3 & 5 \end{pmatrix},$$

and use the result to solve the system of equations:

$$3x + 2y + 4z = 7,$$

$$2x + y + z = 7,$$

$$x + 3y + 5z = 2.$$

(2) Solve the system

$$x + y + z = 3,$$

$$2x - y + 3z = 16,$$

$$3x + y - z = -3,$$

by using Cramer's rule.

(3) Use Gauss elimination method to solve the system

$x_1 + x_2 - 2x_3 = 3,$

$4x_1 - 2x_2 + x_3 = 5,$

$3x_1 - x_2 + 3x_3 = 8.$

(4) Solve the system

$10x_1 + x_2 + x_3 = 12,$

$x_1 + 10x_2 + x_3 = 12,$

$x_1 + x_2 + 10x_3 = 12,$

by using the Gauss-Jordan elimination method.

(5) Factorize the matrix

$$\begin{pmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{pmatrix}$$

in the form LU, where L is a lower triangular matrix, and U is an upper triangular matrix such that the diagonal elements of U are 1. Use this result to solve the system

$5x - 2y + z = 4$

$7x + y - 5z = 8$

$3x + 7y + 4z = 10.$

(6) Solve the systems given in the previous problems by the other methods.

CHAPTER- 3

Interpolation

We say that a polynomial $p(x)$ interpolates a real valued function $f(x)$ at the points $x_0, x_1, \dots, x_n$, if $p(x_0) = f(x_0), p(x_1) = f(x_1), p(x_2) = f(x_2), \dots\dots,$ $p(x_n) = f(x_n)$.

Theorem:

Given a real valued function $f(x)$ and $n+1$ distinct points $x_0, x_1, \dots, x_n$, there exists exactly one polynomial of degree $\leq n$ which interpolates $f(x)$ at $x_0, x_1, \dots, x_n$.

Proof: If there were two different such polynomials $p_1(x)$ and $p_2(x)$, then $p_1(x) - p_2(x)$ would be a polynomial of degree $\leq n$ with $n+1$ distinct roots $x_0, x_1, \dots, x_n$. This is impossible. This proves the uniqueness part. Now, let us prove the existence part. The required polynomial is the Lagrange's interpolation polynomial:

$$p_n(x) = \sum_{1=0}^{n} \frac{(x - x_0)(x - x_1) \dots (x - x_{i-1})(x - x_{i+1}) \dots (x - x_n)}{(x_i - x_0)(x_i - x_1) \dots (x_i - x_{i-1})(x_i - x_{i+1}) \dots (x_i - x_n)} f(x_i)$$

(Verification is left). This completes the proof.

Suppose that the values $f_0, f_1, \dots, f_n$ of $f(x)$ are given at the distinct points $x_0, x_1, \dots, x_n$. That is, $f(x_i) = f_i$ are given for $i = 0, 1, \dots, n$. Construct the polynomial $p_n(x)$ given in the proof of the previous theorem. Since $f(x_i) = p_n(x_i)$ for all i, we use the formula $f(x) \approx p_n(x)$ to evaluate $f(x)$, for any given $x \neq x_i$, for all i. This is the fundamental idea in the study of interpolating polynomials under the heading "Interpolation".

Lagrange's interpolation formula:

If a real valued function $f(x)$ gives the values $f(x_0) = f_0, f(x_1) = f_1, \dots, f(x_n) = f_n$ at the $(n+1)$ points $x_0, x_1, \dots, x_n$, then the Lagrange's interpolating polynomial which interpolates the function $f(x)$ at the points $x_0, x_1, \dots, x_n$, or which fits the data $(x_0, f_0), (x_1, f_1), \dots, (x_n, f_n)$ is

$$p_n(x) = \sum_{1=0}^{n} \frac{(x - x_0)(x - x_1) \dots (x - x_{i-1})(x - x_{i+1}) \dots (x - x_n)}{(x_i - x_0)(x_i - x_1) \dots (x_i - x_{i-1})(x_i - x_{i+1}) \dots (x_i - x_n)} f_i \ .$$

Example:

Given the following data, evaluate $f(3)$ by using Lagrange's interpolating polynomial.

x	1	2	5
$f(x)$	1	4	10

Solution: The Lagrange's interpolating polynomial is

$$p_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2).$$

That is,

$$p_2(x) = \frac{(x-2)(x-5)}{(1-2)(1-5)} \times 1 + \frac{(x-1)(x-5)}{(2-1)(2-5)} \times 4 + \frac{(x-1)(x-2)}{(5-1)(5-2)} \times 10.$$

Therefore,

$$f(3) \approx p_2(3)$$

$$= \frac{(3-2)(3-5)}{(1-2)(1-5)} \times 1 + \frac{(3-1)(3-5)}{(2-1)(2-5)} \times 4 + \frac{(3-1)(3-2)}{(5-1)(5-2)} \times 10$$

$$\approx 6.4999.$$

Thus, $f(3) \approx 6.4999.$

Example:

Find Lagrange's interpolating polynomial fitting the points $y(1) = -3, y(3) = 0, y(4) = 30, y(6) = 132$. Hence, find $f(5)$.

Solution: The table for given data is

x	1	3	4	6
$y = y(x)$	-3	0	30	132

The Lagrange's interpolating polynomial that fits this data is $p_3(x) = \frac{(x-3)(x-4)(x-6)}{(1-3)(1-4)(1-6)} \times (-3) + \frac{(x-1)(x-4)(x-6)}{(3-1)(3-4)(3-6)} \times 0 + \frac{(x-1)(x-3)(x-6)}{(4-1)(4-3)(4-6)} \times 30 + \frac{(x-1)(x-3)(x-4)}{(6-1)(6-3)(6-4)} \times 132 = \frac{1}{2}(-x^3 + 27x^2 - 92x + 60)$, on simplification. Now, $y(5) \approx p_3(5) = \frac{1}{2}(-5^3 + 27 \times 5^2 - 92 \times 5 + 60) = 75$. Thus, $y(5) \approx 75$.

Remark:

Note that $f(x_i) = p_n(x_i)$, for the Lagrange's interpolating polynomial $p_n(x), 1 = 0, 1, 2, \ldots, n$. By the theorem, such a polynomial is unique. We shall discuss the other forms of this polynomial. The other forms are introduced for computational conveniences.

Finite difference operators:

We assume that the points $x_0, x_1, \ldots, x_n$ are equispaced by a step length $h > 0$. That is, $x_i = x_0 + ih$. That is, $x_0, x_1, \ldots, x_n$ are $x_0, x_0 + h, x_0 + 2h, \ldots, x_0 + nh$. Consider a set of points (x_i, y_i), $i = 0, 1, 2, \ldots, n$, (or), $(x_i, f(x_i))$, $i = 0, 1, 2, \ldots, n$ with $y_i = f(x_i)$. We define the forward difference operator Δ by $\Delta y_i = y_{i+1} - y_i$. That is, $\Delta y_0 = y_1 - y_0$, $\Delta y_1 = y_2 - y_1$, …., $\Delta y_{n-1} = y_n - y_{n-1}$. We define Δ^2 by $\Delta^2 y_i = \Delta(\Delta y_i) = \Delta(y_{i+1} - y_i) = \Delta(y_{i+1}) - \Delta(y_i) = (y_{i+2} - y_{i+1}) - (y_{i+1} - y_i) = y_{i+2} - 2y_i + y_i$. In general, $\Delta^r y_i = \Delta(\Delta^{r-1} y_i)$.

We have the following forward difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
x_0	y_o								
		Δy_0							
x_1	y_1		$\Delta^2 y_0$						
		Δy_1		$\Delta^3 y_0$					
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$				
		Δy_2		$\Delta^3 y_1$					
x_3	y_3		$\Delta^2 y_2$		$\vdots$				
		Δy_3		$\vdots$					
x_4	y_4		$\vdots$		$\vdots$				
		$\vdots$		$\vdots$					
$\vdots$	$\vdots$		$\vdots$		$\vdots$				
		$\vdots$		$\vdots$					

The middle placements and the definitions give computational procedure.

We define the backward difference operator ∇ by $\nabla y_i = y_i - y_{i-1}$. That is, $\nabla y_n = y_n - y_{n-1}$, $\nabla y_{n-1} = y_{n-1} - y_{n-2}$, ….., $\nabla y_1 = y_1 - y_0$. We define ∇^2 by $\nabla^2 y_i = \nabla(\nabla y_i) = \nabla(y_i - y_{i-1}) = \nabla y_i - \nabla y_{i-1} = (y_i - y_{i-1}) - (y_{i-1} - y_{i-2}) = y_i - 2y_{i-1} + y_{i-2}$. In general, $\nabla^r y_i = \nabla(\nabla^{r-1} y_i)$.

We have the following backward difference table:

$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{\nabla y}$	$\boldsymbol{\nabla^2 y}$	$\boldsymbol{\nabla^3 y}$	$\boldsymbol{\nabla^4 y}$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
		$\vdots$		$\vdots$					
$\vdots$	$\vdots$		$\vdots$		$\vdots$				
		$\vdots$		$\vdots$					
x_{n-4}	y_{n-4}		$\vdots$		$\vdots$				
		∇y_{n-3}		$\vdots$					
x_{n-3}	y_{n-3}		$\nabla^2 y_{n-2}$		$\vdots$				
		∇y_{n-2}		$\nabla^3 y_{n-1}$					
x_{n-2}	y_{n-2}		$\nabla^2 y_{n-1}$		$\nabla^4 y_n$				
		∇y_{n-1}		$\nabla^3 y_n$					
x_{n-1}	y_{n-1}		$\nabla^2 y_n$						
		∇y_n							
x_n	y_n								

The middle placements and the definitions give computational procedure.

Example:

Construct a forward difference table and a backward difference table for the following values of x and y.

$\boldsymbol{x}$	0.1	0.3	0.5	0.7	0.9
$\boldsymbol{y}$	0.003	0.067	0.148	0.248	0.370

Solution: Note that $h = 0.2$.

The forward difference table is:

$\boldsymbol{x}$	$\boldsymbol{y}$	$\Delta\boldsymbol{y}$	$\Delta^2\boldsymbol{y}$	$\Delta^3\boldsymbol{y}$	$\Delta^4\boldsymbol{y}$
0.1	0.003				
		0.064			
0.3	0.067		0.017		
		0.081		0.002	
0.5	0.148		0.019		0.001
		0.100		0.003	
0.7	0.248		0.022		
		0.122			
0.9	0.370				

The backward difference table is:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
0.1	0.003				
		0.064			
0.3	0.067		0.017		
		0.081		0.002	
0.5	0.148		0.019		0.001
		0.100		0.003	
0.7	0.248		0.022		
		0.122			
0.9	0.370				

Remark: We have the same table, because $\Delta y_i = y_{i+1} - y_i = \nabla y_{i+1}$. The top values 0.1, 0.003, 0.064, 0.017, 0.002, 0.001 in the first table refer to $x_0,\ y_0, \Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \Delta^4 y_0$. The same values in the second table refer to $x_0,\ y_0, \nabla y_1, \nabla^2 y_2, \nabla^3 y_3, \nabla^4 y_4$. The bottom values 0.9, 0.370, 0.122, 0.022, 0,003, 0.001 in the second table refer to $x_4,\ y_4, \nabla y_4, \nabla^2 y_4, \nabla^3 y_4, \nabla^4 y_4$. The same values in the first table refer to $x_4,\ y_4, \Delta y_3, \Delta^2 y_2, \Delta^3 y_1, \Delta^4 y_0$.

Newton's forward difference interpolation formula:

Suppose that the values of a function $f(x)$ are given at $x_0, x_0 + h, x_0 + 2h, \dots., x_0 + nh$. Write $x_i = x_0 + ih$, and $y_i = f(x_i) = f(x_0 + ih)$ for

$i = 0, 1, \dots, n$. Write $x = x_0 + ph$ with $p = \frac{x - x_0}{h}$. Then, Newton's forward difference interpolation formula is:

$$
\begin{aligned}
f(x) &= f(x_0 + ph) \\
&\approx f(x_0) + \frac{p}{1!}\Delta f(x_0) + \frac{p(p-1)}{2!}\Delta^2 f(x_0) \\
&+ \frac{p(p-1)(p-2)}{3!}\Delta^3 f(x_0) + \cdots \dots \\
&+ \frac{p(p-1)(p-2)\dots(p-(n-1))}{n!}\Delta^n f(x_0).
\end{aligned}
$$

This formula can be applied to evaluate $f(x)$ for any x. (However, it is applied in some books to evaluate $f(x)$ only when x lies between x_0 and $x_0 + h$). The corresponding polynomial in x,

$$
\begin{aligned}
p_n(x) &= p_n(x_0 + ph) \\
&= f(x_0) + \frac{p}{1!}\Delta f(x_0) + \frac{p(p-1)}{2!}\Delta^2 f(x_0) \\
&+ \frac{p(p-1)(p-2)}{3!}\Delta^3 f(x_0) + \cdots \dots \\
&+ \frac{p(p-1)(p-2)\dots(p-(n-1))}{n!}\Delta^n f(x_0)
\end{aligned}
$$

obtained after replacing p by $\frac{x - x_0}{h}$ is called the Newton's forward difference interpolating polynomial. Then $p_n(x)$ fits the data (x_i, y_i), $i = 0, 1, \dots, n$ (Proof is omitted). Both of them may be applied to evaluate $f(x)$.

Example: Evaluate $f(15)$, given the following table of values.

$\boldsymbol{x}$	10	20	30	40	50
$\boldsymbol{y = f(x)}$	46	66	81	93	101

Solution: The forward difference table is:

$\boldsymbol{x}$	$\boldsymbol{y}$	$\Delta \boldsymbol{y}$	$\Delta^2 \boldsymbol{y}$	$\Delta^3 \boldsymbol{y}$	$\Delta^4 \boldsymbol{y}$
10	46				
		20			
20	66		-5		
		15		2	
30	81		-3		-3
		12		-1	
40	93		-4		
		8			
50	101				

Here $x_0 = 10, h = 10, x = x_0 + ph = 15$, and $p = \frac{15-10}{10} = 0.5$. Also, $y_0 = 46, \Delta y_0 = 20, \Delta^2 y_0 = -5, \Delta^3 y_0 = 2, \Delta^4 y_0 = -3$. The Newton's forward difference formula is

$$f(x) = f(x_0 + ph) \approx y_0 + \frac{p}{1!}\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0.$$

Therefore, we have $f(15) \approx 46 + (0.5) \times 20 + \frac{0.5(0.5-1)}{2} \times (-5) +$

$\frac{0.5(0.5-1)(0.5-2)}{6} \times 2 + \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{24} \times (-3) \approx 56.8672$. Thus, we have

$f(15) = 56.8672$.

Example: Find Newton's forward difference interpolating polynomial for the following data:

x	0.1	0.2	0.3	0.4	0.5
$y = f(x)$	1.40	1.56	1.76	2.00	2.28

Use it to evaluate $f(0.15)$.

Solution: The forward difference table is:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.1	1.40				
		0.16			
0.2	1.56		0.04		
		0.20		0	
0.3	1.76		0.04		0
		0.24		0	
0.4	2.00		0.04		
		0.28			
0.5	2.28				

Here $x_0 = 0.1$, $h = 0.1$, $y_0 = 1.40$, $\Delta y_0 = 0.16$, $\Delta^2 y_0 = 0.04$, and $\Delta^3 y_0 = 0 = \Delta^4 y_0$. Also, $x = x_0 + ph$ and $p = \frac{x - x_0}{h} = \frac{x - 0.1}{0.1}$. The Newton forward interpolating polynomial is

$f(x) = f(x_0 + ph) \approx y_0 + \frac{p}{1!}\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$ with $p = \frac{x-0.1}{0.1}$. This polynomial reduces to $1.4 + \frac{x-0.1}{0.1} \times 0.16 + \frac{1}{2}(10x - 1)(10x - 1 - 1) \times 0.04 = 2x^2 + x + 1.28$. Thus, the required polynomial is $2x^2 + x + 1.28$.

Moreover, $f(0.15) \approx 2 \times 0.15^2 + 0.15 + 1.28 = 0.045 + 0.15 + 1.28 = 1.475$.

Thus, $f(0.15) \approx 1.475$.

Newton's backward difference interpolation formula:

Suppose that the values of a function $f(x)$ are given at $x_0, x_0 + h, \dots, x_0 + nh$. Write $x_i = x_0 + ih$, and $y_i = f(x_i) = f(x_0 + ih)$ for $i = 0, 1, 2, \dots, n$.

Write $x = x_n + ph$ with $p = \frac{x - x_n}{h}$. Then, Newton's backward difference interpolation formula is

$$f(x) = f(x_n + ph) \approx$$

$$f(x_n) + \frac{p}{1!}\nabla f(x_n) + \frac{p(p+1)}{2!}\nabla^2 f(x_n) + \frac{p(p+1)(p+2)}{3!}\nabla^3 f(x_n) + \cdots \dots + \frac{p(p+1)(p+2)\dots(p+(n-1))}{n!}\nabla^n f(x_n).$$

This formula can be applied to evaluate $f(x)$ for any x. (However, it is applied in some books to evaluate $f(x)$ only when x lies between x_n and $x_n - h$). The corresponding polynomial in x

$$p_n(x) = p_n(x_n + ph)$$

$$\approx f(x_n) + \frac{p}{1!}\nabla f(x_n) + \frac{p(p+1)}{2!}\nabla^2 f(x_n)$$

$$+ \frac{p(p+1)(p+2)}{3!}\nabla^3 f(x_n) + \cdots \ldots$$

$$+ \frac{p(p+1)(p+2)\ldots(p+(n-1))}{n!}\nabla^n f(x_n)$$

obtained after replacing p by $\frac{x-x_n}{h}$ is called the Newton's backward difference interpolating polynomial. Then $p_n(x)$ fits the data (x_i, y_i), $i = 0, 1, \ldots, n$. Both of them may be applied to evaluate $f(x)$.

Example:

For the following table of values, estimate $f(7.5)$. Use Newton's backward difference formula.

$\boldsymbol{x}$	1	2	3	4	5	6	7	8	
$\boldsymbol{y = f(x)}$	1	8	27	64	125	216	343	512	

Solution: The backward difference table is

$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{\nabla y}$	$\boldsymbol{\nabla^2 y}$	$\boldsymbol{\nabla^3 y}$	$\boldsymbol{\nabla^4 y}$
1	1				
		7			
2	8		12		
		19		6	
3	27		18		0
		37		6	
4	64		24		0
		61		6	
5	125		30		0
		91		6	
6	216		36		0
		127		6	
7	343		42		
		169			
8	512				

Here $n = 7, x_n = 8 = x_7$, $h = 1, y_n = 512 = f(x_n), \nabla y_n = 169, \nabla^2 y_n = 42,$

$\nabla^3 y_n = 6, \nabla^4 y_n = \nabla^5 y_n = \nabla^6 y_n = \nabla^7 y_n = 0$. Also, $x = 7.5 = x_n + ph,$

$p = \frac{x-x_n}{h} = \frac{7.5-8}{1} = -0.5$. Since $\nabla^i y_n = 0$ for $i = 4, 5, 6, 7$, the Newton backward difference formula becomes

$f(x) = f(x_n + ph) \approx y_n + \frac{p}{1!}\nabla \mathrm{y_n} + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$. Therefore,

$f(7.5) \approx 512 + (-0.5) \times 169 + \frac{(-0.5)(-0.5+1)}{2} \times 42 + \frac{(-0.5)(-0.5+1)(-0.5+2)}{6} \times 6 = 421.875$. Thus, $f(7.5) \approx 421.875$.

Example: Find Newton's backward difference interpolating polynomial for the following data:

$\boldsymbol{x}$	0.1	0.2	0.3	0.4	0.5
$\boldsymbol{y = f(x)}$	1.40	1.56	1.76	2.00	2.28

Hence, find $f(0.45)$.

Solution: The backward difference table is:

$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{\nabla y}$	$\boldsymbol{\nabla^2 y}$	$\boldsymbol{\nabla^3 y}$	$\boldsymbol{\nabla^4 y}$
0.1	1.40				
		0.16			
0.2	1.56		0.04		
		0.20		0	
0.3	1.76		0.04		0
		0.24		0	
0.4	2.00		0.04		

		0.28			
0.5	2.28				

Here, $n = 4, x_n = 0.5, h = 0.1, y_n = 2.28, \nabla y_n = 0.28, \nabla^2 y_n = 0.04, \nabla^3 y_n = \nabla^4 y_n = 0$. Also, $x = x_n + ph, p = \frac{x - x_n}{h} = \frac{x - 0.5}{0.1} = 10x - 5$. The Newton's backward interpolating polynomial is

$y_n + \frac{p}{1!}\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n$ with $p = 10x - 5$. This reduces to $2.28 + (10x - 5) \times 0.28 + 0.5(10x - 5)(10x - 5 + 1) \times 0.04 = 2x^2 + x + 1.28$. Thus, the required polynomial is $2x^2 + x + 1.28$. Moreover, $f(0.45) \approx 2 \times 0.45^2 + 0.45 + 1.28 = 0.405 + 0.45 + 1.28 = 2.135$. Thus, $f(0.45) \approx 2.135$.

Remark:

The second examples given in the section of Newton forward difference interpolation formula and in the section of Newton backward difference interpolation formula lead to the same polynomial $2x^2 + x + 1.28$. This agrees with the theorem proved in the beginning of this chapter.

Example:

Show that (i) the Newton forward interpolating polynomial, (ii) the Newton backward interpolating polynomial, and (iii) the Lagrange's interpolating polynomial, which interpolate the values 1.40, 1.56, 1.76, 2.00, 2.28 at the points

0.1, 0.2, 0.3, 0.4, 0.5 (or, which fits the data (0.1, 1.4), (0.2, 1.56), (0.3, 1.76), (0.4, 2), (0.5, 2.28)), which have degree ≤ 4 are equal to $2x^2 + x + 1.28$.

Solution: We have already seen that the Newton forward interpolating polynomial and the Newton backward interpolating polynomial are $2x^2 + x + 1.28$. So, they will not be shown here again. The Lagrange's interpolating polynomial is

$$\frac{(x-0.2)(x-0.3)(x-0.4)(x-0.5)}{(0.1-0.2)(0.1-0.3)(0.1-0.4)(0.1-0.5)} \times 1.4 + \frac{(x-0.1)(x-0.3)(x-0.4)(x-0.5)}{(0.2-0.1)(0.2-0.3)(0.2-0.4)(0.2-0.5)} \times 1.56 +$$

$$\frac{(x-0.1)(x-0.2)(x-0.4)(x-0.5)}{(0.3-0.1)(0.3-0.2)(0.3-0.4)(0.3-0.5)} \times 1.76 + \frac{(x-0.1)(x-0.2)(x-0.3)(x-0.5)}{(0.4-0.1)(0.4-0.2)(0.4-0.3)(0.4-0.5)} \times 2 +$$

$$\frac{(x-0.1)(x-0.2)(x-0.3)(x-0.4)}{(0.5-0.1)(0.5-0.2)(0.5-0.3)(0.5-0.4)} \times 2.28 = 2x^2 + x + 1.28 \text{ (on simplification).}$$

Central difference formulae:

Gauss interpolation formulae:

Consider a set of equispaced points $\dots\dots, x_{-3},\ x_{-2}, x_{-1}, x_0,\ x_1,$ $x_2, x_3, \dots\dots$ with step length $h > 0$. That is, $x_i = x_0 + h$ for $i = 0, 1, -1, 2, -2, \dots\dots$. That is, $x_1 = x_0 + h$, $x_{-1} = x_0 - h$, $x_2 = x_0 + 2h$, $x_{-2} = x_0 - 2h$, ……… . Suppose that the function values $f(x_i) = y_i$ of a function $f(x)$ are given at these points. Then, we have the difference table:

$\boldsymbol{x}$	$\boldsymbol{y}$	$\Delta\boldsymbol{y}$	$\Delta^2\boldsymbol{y}$	$\Delta^3\boldsymbol{y}$	$\Delta^4\boldsymbol{y}$	$\cdots$	$\cdots$
$\vdots$	$\vdots$		$\vdots$		$\vdots$		
		$\vdots$		$\vdots$			

x_{-2}	y_{-2}		$\vdots$		$\vdots$		
		Δy_{-2}		$\vdots$			
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		$\vdots$		
		Δy_{-1}		$\Delta^3 y_{-2}$			
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$	$\cdots$	$\cdots$
		Δy_0		$\Delta^3 y_{-1}$			
x_1	y_1		$\Delta^2 y_0$		$\vdots$		
		Δy_1		$\vdots$			
x_2	y_2		$\vdots$		$\vdots$		
		$\vdots$		$\vdots$			
$\vdots$	$\vdots$		$\vdots$		$\vdots$		

Write $x = x_0 + ph$. Then the Gaussian forward interpolation formula is

$$f(x) = f(x_0 + ph)$$

$$= y_0 + \frac{p}{1!}\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2}$$

$$+ \frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\Delta^5 y_{-2} + \cdots \ldots$$

The necessary positions in the table are the followings:

y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$	
	Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$		$\Delta^7 y_{-3}$

The Gaussian backward interpolation formula is

$$f(x) = f(x_0 + ph)$$

$$= y_0 + \frac{p}{1!}\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2}$$

$$+ \frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2}$$

$$+ \frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\Delta^5 y_{-3}$$

$$+ \frac{(p+3)(p+2)(p+1)p(p-1)(p-2)}{6!}\Delta^6 y_{-3} + \cdots ....$$

The necessary positions in the table are the followings:

	Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$		$\Delta^7 y_{-4}$
y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$	

Use as many terms as possible from the table for calculations. (First one is applied in some books when x lies between x_0 and x_1 and the second one is applied when x lies between x_{-1} and x_0).

We follow these notations to write the next three formulae: (i) Stirling's formula, (ii) Bessel's formula, and (iii) Everett's formula.

Stirling's formula:

$$f(x) = f(x_0 + ph)$$

$$= y_0 + \frac{p}{1!}\frac{(\Delta y_0 + \Delta y_{-1})}{2} + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\frac{(\Delta^3 y_{-1} + \Delta^3 y_{-3})}{2}$$

$$+ \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \cdots \ldots.$$

This is the mean of the Gauss's forward and backward formulae.

Bessel's formula:

$$f(x) = f(x_0 + ph)$$

$$= y_0 + \frac{p}{1!}\Delta y_0 + \frac{p(p-1)}{2!}\frac{(\Delta^2 y_{-1} + \Delta^2 y_0)}{2} + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\frac{(\Delta^4 y_{-2} + \Delta^4 y_{-1})}{2} + \cdots \ldots.$$

Everett's formula:

$$f(x) = f(x_0 + ph)$$

$$= \left\{ q y_0 + \frac{q(q^2-1^2)}{3!}\Delta^2 y_{-1} + \frac{q(q^2-1^2)(q^2-2^2)}{5!}\Delta^4 y_{-2} + \cdots \ldots \right\}$$

$$+ \left\{ p y_{-1} + \frac{p(p^2-1^2)}{3!}\Delta^2 y_0 + \frac{p(p^2-1^2)(p^2-2^2)}{5!}\Delta^4 y_{-1} + \cdots \ldots \right\}$$

where $q = 1 - p$.

Remark:

When these formulae are applied, the point x_0 is chosen such that we use as many as terms as possible.

Example:

From the following table, find the value of $e^{1.17}$.

x	1.00	1.05	1.10	1.15	1.20	1.25	1.30
e^x	2.7183	2.8577	3.0042	3.1582	3.3201	3.4903	3.6693

Use

(a) Gauss's forward formula

(b) Gauss's backward formula

(c) Stirling's interpolation formula

(d) Bessel's interpolation formula

(e) Everett's interpolation formula.

Solution: Take $y = f(x) = e^x$. The difference table is given below.

x	e^x	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
1.00	2.7183						
		0.1394					
1.05	2.8577		0.0071				
		0.1465		0.0004			

1.10	3.0042		0.0075		0		
		0.1540		0.0004		0	
1.15	3.1582		0.0079		0		0.0001
		0.1619		0.0004		0.0001	
1.20	3.3201		0.0083		0.0001		
		0.1702		0.0005			
1.25	3.4903		0.0088				
		0.1790					
1.30	3.6693						

(a) Gauss's forward interpolation formula is

$$f(x) = f(x_0 + ph)$$

$$\approx y_0 + \frac{p}{1!}\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2}$$

$$+ \frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\Delta^5 y_{-2}$$

$$+ \frac{(p+2)(p+1)p(p-1)(p-2)(p-3)}{6!}\Delta^6 y_{-3} \; .$$

Let us take $x_0 = 1.15$. Then $x = 1.17 = 1.15 + p(0.05)$ and $p = \frac{1.17-1.15}{0.05} = \frac{2}{5}$.

$y_0 = 3.1582$, $\Delta y_0 = 0.1619$, $\Delta^2 y_{-1} = 0.0079$, $\Delta^3 y_{-1} = 0.0004$, $\Delta^4 y_{-2} = 0$,

$\Delta^5 y_{-2} = 0.0001$, $\Delta^6 y_{-3} = 0.0001$. Thus, we have $f(1.17) = e^{1.17} \approx 3.1582 +$

$\frac{2}{5} \times 0.1619 + \frac{\frac{2}{5}(\frac{2}{5}-1)}{2} \times 0.0079 + \frac{\left(\frac{2}{5}+1\right)\frac{2}{5}(\frac{2}{5}-1)}{6} \times 0.0004 + 0 +$

$\frac{\left(\frac{2}{5}+2\right)\left(\frac{2}{5}+1\right)\frac{2}{5}\left(\frac{2}{5}-1\right)(\frac{2}{5}-2)}{120} \times 0.0001 + \frac{\left(\frac{2}{5}+2\right)\left(\frac{2}{5}+1\right)\frac{2}{5}\left(\frac{2}{5}-1\right)\left(\frac{2}{5}-2\right)(\frac{2}{5}-3)}{720} \times 0.0001 \approx 3.1582 +$

$0.0648 - 0.0009 = 3.2221$. Thus, we have $e^{1.17} \approx 3.2221$.

(b) Gauss's backward interpolation formula is

$f(x) = f(x_0 + ph) \approx y_0 + \frac{p}{1!}\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} +$

$\frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2} + \frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\Delta^5 y_{-3} +$

$\frac{(p+3)(p+2)(p+1)p(p-1)(p-2)}{6!}\Delta^6 y_{-3}$.

Let us take $x = 1.15$. Then $x = 1.17 = 1.15 + p(0.05)$ and $p = \frac{2}{5}$. $y_0 =$

$3.1582, \Delta y_{-1} = 0.1540, \Delta^2 y_{-1} = 0.0079, \Delta^3 y_{-2} = 0.0004, \Delta^4 y_{-2} = 0,$

$\Delta^5 y_{-3} = 0, \Delta^6 y_{-3} = 0.0001$. Thus, we have

$f(1.17) = e^{1.17}$

$\approx 3.1582 + \frac{2}{5} \times 0.1540 + \frac{(\frac{2}{5}+1)\frac{2}{5}}{2} \times 0.0079 + \frac{\left(\frac{2}{5}+1\right)\frac{2}{5}(\frac{2}{5}-1)}{6} \times 0.0004 +$

$practically\ zero\ terms$

$\approx 3.1582 + 0.0616 + 0.0022 + 0\ (correct\ to\ 4\ decimals) = 3.2220.$

Thus, we have $e^{1.17} \approx 3.2220$.

(c) Stirling's interpolation formula is

$f(x) = f(x_0 + ph) \approx y_0 + \frac{p}{1!}\frac{(\Delta y_0 + \Delta y_{-1})}{2} + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\frac{(\Delta^3 y_{-1} + \Delta^3 y_{-3})}{2} +$

$\frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2}$.

Let us take $x_0 = 1.15$. Then $x = 1.17 = 1.15 + p(0.05)$ and $p = \frac{1.17-1.15}{0.05} = \frac{2}{5}$.

$y_0 = 3.1582$, $\Delta y_0 = 0.1619$, $\Delta y_{-1} = 0.1540$, $\Delta^2 y_{-1} = 0.0079$, $\Delta^3 y_{-1} = 0.0004$,

$\Delta^3 y_{-3} = 0.0004$, $\Delta^4 y_{-2} = 0$. Therefore, $f(1.17) = e^{1.17}$

$\approx 3.1582 + \frac{2}{5} \times \frac{0.1619+0.1540}{2} + \frac{1}{2} \times \left(\frac{2}{5}\right)^2 \times 0.0079 + \frac{1}{6} \times \frac{2}{5}\left(\frac{4}{25} - 1\right) \times$

$\frac{0.0004+0.0004}{2} + 0$

$\approx 3.1582 + 0.0632 + 0.0006 - 0$ $(correct\ to\ 4\ decimals) = 3.2220$. Thus, we have $e^{1.17} \approx 3.2220$.

(d) Bessel's interpolation formula is

$f(x) = f(x_0 + ph) \approx y_0 + \frac{p}{1!}\Delta y_0 + \frac{p(p-1)}{2!}\frac{(\Delta^2 y_{-1} + \Delta^2 y_0)}{2} + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_{-1} +$

$\frac{(p+1)p(p-1)(p-2)}{4!}\frac{(\Delta^4 y_{-2} + \Delta^4 y_{-1})}{2}$.

Let us take $x = 1.15$. Then $x = 1.17 = 1.15 + p(0.05)$ and $p = \frac{2}{5}$. $y_0 = 3.1582$, $\Delta y_0 = 0.1619$, $\Delta^2 y_0 = 0.0083$, $\Delta^2 y_{-1} = 0.0079$, $\Delta^3 y_{-1} = 0.0004$, $\Delta^4 y_{-1} = 0.0001$, $\Delta^4 y_{-2} - 0$. Therefore, $f(1.17) = e^{1.17}$

$\approx 3.1582 + \frac{2}{5} \times 0.1619 + \frac{\frac{2}{5}(\frac{2}{5}-1)}{2} \times \frac{0.0079+0.0083}{2} + \frac{\frac{2}{5}\left(\frac{2}{5}-1\right)(\frac{2}{5}-\frac{1}{2})}{6} \times 0.0004 +$

$practically\ 0$

$\approx 3.1582 + 0.0648 - 0.0010 + 0 \ \ (correct\ to\ 4\ decimals) = 3.2220.$

Thus, we have $e^{1.17} \approx 3.2220.$

(e) Everett's interpolation formula is

$f(x) = f(x_0 + ph) \approx \left\{qy_0 + \frac{q(q^2-1^2)}{3!}\Delta^2 y_{-1} + \frac{q(q^2-1^2)(q^2-2^2)}{5!}\Delta^4 y_{-2}\right\} +$

$\left\{py_{-1} + \frac{p(p^2-1^2)}{3!}\Delta^2 y_0 + \frac{p(p^2-1^2)(p^2-2^2)}{5!}\Delta^4 y_{-1}\right\}$, where $q = 1 - p.$

Let us take $x = 1.15$. Then $x = 1.17 = 1.15 + p(0.05)$ and $p = \frac{2}{5}$ and

$q = 1 - p = \frac{3}{5}$. $y_0 = 3.1582, \Delta^2 y_{-1} = 0.0079, \Delta^4 y_{-2} = 0, y_1 = 3.3201,$

$\Delta^2 y_0 = 0.0083, \Delta^4 y_{-1} = 0.0001$. Therefore,

$f(1.17) = e^{1.17}$

$\approx \frac{3}{5} \times 3.1582 + \frac{\frac{3}{5}(\frac{9}{25}-1)}{6} \times 0.0079 + 0 + \frac{2}{5} \times 3.3201 + \frac{\frac{2}{5}(\frac{4}{25}-1)}{6} \times 0.0083 +$

$practically\ 0$

$\approx 1.8949 - 0.0005 + 1.3280 - 0.0004 = 3.2220$. Thus, we have $e^{1.17} \approx$

$3.2220.$

We next give an interpolation formula, which requires function values of its first order derivative at the node points. Newton's forward and backward formulae and the five formulae applied in the previous example are applicable only for equispaced points with a fixed step length h. Lagrange's formula is applicable for the general case. The next Hermite formula is also applicable for the general case in the sense that it does not depend on a fixed step length h.

Hermite interpolation formula:

Suppose that the values of a real valued function $f(x)$ and its derivative $f^{'}(x)$ are given at the $(n+1)$ distinct points $x_0, x_1, \dots, x_n$. Suppose that $f(x_i) = f_i = y_i$ and $f^{'}(x_i) = f_i^{'} = y_i^{'}$ for every i. To each i, write

$$l_i(x) = \frac{(x-x_0)(x-x_1)\dots(x-x_{i-1})(x-x_{i+1})\dots(x-x_n)}{(x_i-x_0)(x_i-x_1)\dots(x_i-x_{i-1})(x_i-x_{i-1})\dots(x_i-x_n)}.$$

Then the Hermite's interpolation formula is

$$f(x) \approx H_{2n+1}(x) = \sum_{i=0}^{n}\big[1-2(x-x_i)l_i^{'}(x_i)\big][l_i(x)]^2 y_i + \sum_{i=0}^{n}(x-x_i)[l_i(x)]^2 y_i^{'}.$$

Let us observe that $H_{2n+1}(x)$ is a (Hermite) polynomial of degree $2n+1$.

Example:

Determine the Hermite polynomial of degree 5 which fits the following data and hence find an approximate value of $\log 2.7$.

x	2.0	2.5	3.0
$y = \log x$	0.69315	0.91629	1.09861
$y^{'} = 1/x$	0.5	0.4	0.33333

Solution: Here $x_0 = 2.0, x_1 = 2.5, x_2 = 3, y_0 = 0.69315, y_1 = 0.91629,$

$y_2 = 1.09861, y_0^{'} = 0.5, y_1^{'} = 0.4, y_2^{'} = 0.33333(= \frac{1}{3}).$

$l_0(x) = \frac{(x-2.5)(x-3)}{(2-2.5)(2-3)} = 2x^2 - 11x + 5;\ l_0^{'}(x) = 4x - 1.$

$l_1(x) = \frac{(x-2)(x-3)}{(2.5-2)(2.5-3)} = -(4x^2 - 20x + 24);\ \ l_1^{'}(x) = -8x + 20.$

$l_2(x) = \frac{(x-2)(x-2.5)}{(3-2)(3-2.5)} = 2x^2 - 9x + 10;\ \ l_2^{'}(x) = 4x - 9.$

The Hermite polynomial of degree 5 is

$H_5(x) = \sum_{i=0}^{2}\left[1 - 2(x - x_i)l_i^{'}(x_i)\right][l_i(x)]^2 y_i + \sum_{i=0}^{2}(x - x_i)[l_i(x)]^2 y_i^{'}$

$= [1 - 2(x - 2)(4 \times 2 - 11)][2x^2 - 11x + 15]^2 \times 0.69315$

$+[1 - 2(x - 2.5)(-8 \times 2.5 + 20)][-(4x^2 - 20x + 24)]^2 \times 0.9162$

$+[1 - 2(x - 3)(4 \times 3 - 9)][2x^2 - 9x + 10]^2 \times 1.09861$

$+(x - 2)[2x^2 - 11x + 15]^2 \times 0.5 + (x - 2.5)[-(4x^2 - 20x + 24)]^2 \times 0.4$

$+(x - 3)[2x^2 - 9x + 10]^2 \times 0.33333.$

If we substitute $x = 2.7$, we get $\log 2.7 \approx H_5(2.7) = 0.993252$. Thus,

$\log 2.7 \approx 0.993252$.

We shall next find another form of Lagrange's form applicable for unevenly spaced points.

Newton's divided difference interpolation formula:

Suppose that the values $f(x_i) = y_i$ of a real valued function $f(x)$ are given at the $(n+1)$ distinct points $x_0, x_1, \dots, x_n$. Then the Newton's divided difference interpolation formula is

$$f(x) \approx p_n(x)$$

$$= f[x_0] + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2]$$

$$+ (x - x_0)(x - x_1)(x - x_3)f[x_0, x_1, x_2, x_3] + \cdots \dots$$

$$+ (x - x_0)(x - x_1) \dots (x - x_{n-1})f[x_0, x_1, \dots, x_n],$$

where the divided differences are

$$f[x_0] = f(x_0),$$

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0},$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0},$$

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0},$$

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$$f[x_0, x_1, \dots, x_n] = \frac{f[x_1, x_2, \dots, x_n] - f[x_0, x_1, \dots, x_{n-1}]}{x_n - x_0}.$$

Remark:

The polynomial $p_n(x)$ is the Lagrange polynomial and hence this polynomial interpolates $f(x)$ at $x_0, x_1, \dots, x_n$. The following form of divided difference table may be applied to evaluate divided differences.

$\boldsymbol{x}$	$\boldsymbol{f}[\,]$	$\boldsymbol{f}[\,,]$	$\boldsymbol{f}[\,,\ ,]$	$\boldsymbol{f}[\,,,,]$	$\cdots$
x_0	$f[x_0]$				
		$f[x_0, x_1]$			
x_1	$f[x_1]$		$f[x_0, x_1, x_2]$		
		$f[x_1, x_2]$		$f[x_0, x_1, x_2, x_3]$	$\cdots$
x_2	$f[x_2]$		$f[x_1, x_2, x_3]$		
		$f[x_2, x_3]$		$\vdots$	
x_3	$f[x_3]$		$\vdots$		
		$\vdots$		$\vdots$	
$\vdots$	$\vdots$		$\vdots$		

Example:

Find the Newton divided difference interpolating polynomial for the following data, and hence evaluate $f(1)$.

x	-1	0	3	6	7
$f(x)$	3	-6	39	822	1611

Solution: The divided difference table is:

$\boldsymbol{x}$	$\boldsymbol{f}[\boldsymbol{x}]$	$\boldsymbol{f}[,]$	$\boldsymbol{f}[,,]$	$\boldsymbol{f}[,,,]$	$\boldsymbol{f}[,,,,]$
-1	3				
		-9			
0	-6		6		
		15		5	
3	39		41		1
		261		13	
6	822		132		
		789			
7	1611				

Here, $x_0 = -1$, $x_1 = 0$, $x_2 = 3$, $x_3 = 6$, $x_4 = 7$, $f[x_0] = 3$, $f[x_0, x_1] = -9$, $f[x_0, x_1, x_2] = 6$, $f[x_0, x_1, x_2, x_3] = 5$, $f[x_0, x_1, x_2, x_3, x_4] = 1$. The required polynomial is

$f[x_0] + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)f[x_0, x_1, x_2, x_3] + (x - x_0)(x - x_1)(x - x_2)(x - x_3)f[x_0, x_1, x_2, x_3, x_4] = 3 + (x + 1) \times (-9) + (x + 1)(x - 0) \times 6 + (x + 1)(x - 0)(x - 3) \times 5 + (x + 1)(x - 0)(x - 3)(x - 6) \times 1 = x^4 - 3x^3 + 5x^2 - 6.$

Therefore, $f(1) \approx 1^4 - 3 \times 1^3 + 5 \times 1^2 - 6 = -3$. Thus, $f(1) \approx -3$.

Final Remarks:

One function g may interpolate another given function f for which the values are known (or given) only at finitely many points $x_0, x_1, \ldots, x_n$. That is, $g(x_i) = f(x_i)$, for every i. One may find many such functions g. Only classical polynomial functions g have been discussed in this chapter for interpolation purpose, because it is easy to work in classical error analysis when polynomial functions are used for interpolations. For practical purposes one may use any function g for which $g(x_i) \approx f(x_i)$, for every i, without considering error analysis. That is, one may apply the relations $f(x) \approx g(x)$, $f'(x) \approx g'(x)$, $f''(x) \approx g''(x)$, … and $\int f(x)dx \approx \int g(x)dx$, for practical purposes.

Exercises:

(1) The following table gives pressure of steam at a given temperature. Using Newton's forward difference interpolation formula, compute the pressure for a temperature of $142°C$.

$Temperature$ $°C$	140	150	160	170	180
$Pressure$ kg/cm^2	3.685	4.854	6.302	8.076	10.225

(2) Find the Newton's backward interpolating polynomial for the following data:

x	1	2	3	4	5
y	1	-1	1	-1	1

(3) Find the interpolating polynomial for the following data using Lagrange's formula. Hence find $f(-2)$.

x	1	2	-4
$y = f(x)$	3	-5	4

(4) Find the interpolating polynomial by Newton's divided difference formula for the following data. Deduce $f(4)$.

x	1	2	3	5

y	0	7	26	124

(5) Using Hermite's interpolation formula, estimate the value of $\log 3.2$ from the following table.

x	3	3.5	4
$y = \log x$	1.09861	1.25276	1.38629
$y' = 1/x$	0.33333	0.28571	0.25000

(6) Find the Hermite polynomial of third degree approximating the function $y(x)$ such that $y(-1) = 1, y(1) = 0, y'(-1) = y'(1) = 0$.

(7) The value of the elliptic integral $K(m) = \int_0^{\pi/2}(1 - m\,sin^2\theta)^{-\frac{1}{2}}\,d\theta$ for certain equidistant values of m are given below. Use Everett's or Bessel's formula to determine $K(0.25)$.

m	0.20	0.22	0.24	0.26	0.28	0.30
$K(m)$	1.659624	1.669850	1.680373	1.691208	1.702374	1.713889

(8) From the following table of values of x and $y = e^x$ interpolate the value of y when $x = 1.91$.

x	1.7	1.8	1.9	2.0	2.1	2.2
$y = e^x$	5.4739	6.0496	6.6859	7.3891	8.1662	9.0250

Use a Gauss formula or Stirling's formula.

CHAPTER- 4

Numerical Differentiation

The heading explains the nature of contents and of the scope of this chapter. There is one more important aspect (application) for this topic. The formulae for numerical differentiation can be applied to solve differential equations.

Numerical differentiation is explained through two inter related approaches. One is based on finite difference operators and another one is based on interpolation formulae. There is one more standard approach.

We already considered two fundamental difference operators Δ and ∇ in the previous chapter for interpolation. Let us introduce two more operators, namely, (forward) shift operator E and central difference operator δ.

Fix a step length $h > 0$, and consider a function $f(x)$. Then $\Delta f(x) = f(x+h) - f(x)$, $\nabla f(x) = f(x) - f(x-h)$, and we define $Ef(x) = f(x+h)$, and $\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$. We shall use: $E^{-1}f(x) = f(x-h)$, $E^{\frac{1}{2}}f(x) = f(x + \frac{h}{2})$, $E^{-\frac{1}{2}}f(x) = f(x - \frac{h}{2})$.

By Taylor's theorem, we have

$$Ef(x) = f(x+h) = f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \cdots \ldots$$

$$= f(x) + \frac{h}{1!}Df(x) + \frac{h^2}{2!}D^2f(x) + \cdots \ldots$$

$$= \left[I + \frac{hD}{1!} + \frac{(hD)^2}{2!} + \frac{(hD)^3}{3!} + \cdots \ldots\right] f(x) = e^{hD}f(x),$$

where $D = \frac{d}{dx}$ and $If(x) = f(x)$.

The series expansions for operators will be meaningful within "our" limitations of applications. So, we need not worry about convergence of any series to be discussed in this chapter. So, we have the fundamental formula: $E = e^{hD}\ (or)\ D = \frac{1}{h}\log E$. This is applicable to derive numerical differentiation formulae through finite difference operators. Note that we have the following relations from the definitions.

(a) $\Delta = E - I;\ \ E = I + \Delta$

(b) $\nabla = I - E^{-1};\ \ E^{-1} = I - \nabla$

(c) $\delta = E^{1/2} - E^{-1/2}$,

where I is the identity operator: $If(x) = f(x)$.

Numerical differentiation using difference operators:

$$D = \frac{1}{h}(\log E) = \frac{1}{h}\log(I + \Delta) = \frac{1}{h}\left(\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \frac{\Delta^5}{5} - \cdots \ldots\right).$$

Therefore,

$$(1)\, f'(x_0) = \frac{1}{h}\left(\Delta f(x_0) - \frac{\Delta^2 f(x_0)}{2} + \frac{\Delta^3 f(x_0)}{3} - \frac{\Delta^4 f(x_0)}{4} + \frac{\Delta^5 f(x_0)}{5} - \cdots \ldots\right)$$

$$D^2 = \frac{1}{h^2}\left(\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \cdots \ldots\right)^2 = \frac{1}{h^2}\left(\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 - \frac{5}{6}\Delta^5 + \cdots \ldots\right) \quad .$$

Therefore,

$$(2)\, f''(x) = \frac{1}{h^2}\left(\Delta^2 f(x_0) - \Delta^3 f(x_0) + \frac{11}{12}\Delta^4 f(x_0) - \frac{5}{6}\Delta^5 f(x_0) + \cdots \ldots\right).$$

Similarly,

$D = \frac{1}{h}\log E = \frac{1}{h}\log(E^{-1})^{-1} = \frac{1}{h}\log(I - \nabla)^{-1} = -\frac{1}{h}\log(I - \nabla) = -\frac{1}{h}\left(\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} + \cdots\right)$ yields

$$(3)\, f'(x_n) = \frac{1}{h}\left(\nabla f(x_n) + \frac{\nabla^2 f(x_n)}{2} + \frac{\nabla^3 f(x_n)}{3} + \frac{\nabla^4 f(x_n)}{4} + \cdots \ldots\right)$$

and

$$(4)\, f''(x_n) = \frac{1}{h^2}\left(\nabla^2 f(x_n) + \nabla^3 f(x_n) + \frac{11}{12}\nabla^4 f(x_n) + \frac{5}{6}\nabla^5 f(x_n) + \cdots \ldots\right).$$

Similarly,

$\delta = E^{1/2} - E^{-1/2} = e^{hD/2} - e^{-hD/2} = 2\sinh\frac{hD}{2}$, $\frac{hD}{2} = \sinh^{-1}\frac{\delta}{2}$,

$D = \frac{2}{h}\sinh^{-1}\frac{\delta}{2} = \frac{1}{h}\left(\delta - \frac{1}{24}\delta^3 + \frac{3}{640}\delta^5 - \cdots \ldots\right)$, $D^2 = \frac{1}{h^2}\left(\delta^2 - \frac{1}{12}\delta^4 + \frac{1}{90}\delta^6 - \cdots \ldots\right)$.

Hence,

$$(5)\, f'(x) = \frac{1}{h}\left(\delta f(x) - \frac{1}{24}\delta^3 f(x) + \frac{3}{640}\delta^5 f(x) - \cdots \ldots\right)$$

and

(6) $f''(x) = \frac{1}{h^2}\left(\delta^2 f(x) - \frac{1}{12}\delta^4 f(x) + \frac{1}{19}\delta^6 f(x) - \cdots \ldots\right).$

The formulae (1), (3) and (5) are used to find first order derivatives, and (2), (4) and (6) are used to find second order derivatives.

Suppose $x_i = x_0 + ih,\ i = 0, 1, \ldots, n,$ with $h > 0$. Then (1) and (2) are applied for x_0; (3) and (4) are applied for x_n, and (5) and (6) are applied when x lies in between (middle of) x_0 and x_n. We use as many terms as possible from the series (1), (2), (3), (4), (5), and (6) to evaluate values for derivatives.

Example:

Compute $f''(0)$ and $f'(0.2)$ from the following tabular data.

x	0.0	0.2	0.4	0.6	0.8	1.0
$f(x)$	1.00	1.16	3.56	13.96	41.96	101.00

Solution: Since $x = 0$ and $x = 0.2$ appear at beginning and near beginning of the table, it is appropriate to use formulae based on forward differences to find the derivative values. The forward difference table is

$\boldsymbol{x}$	$\boldsymbol{f(x)}$	$\Delta \boldsymbol{f(x)}$	$\Delta^2 \boldsymbol{f(x)}$	$\Delta^3 \boldsymbol{f(x)}$	$\Delta^4 \boldsymbol{f(x)}$	$\Delta^5 \boldsymbol{f(x)}$
0.0	1.00					
		0.16				
0.2	1.16		2.24			
		2.40		5.76		

0.4	3.56		8.00		3.84	
		10.40		9.60		0
0.6	13.96		17.60		3.84	
		28.00		13.44		
0.8	41.96		31.04			
		59.04				
1.0	101.00					

Here $h = 0.2$.

$f'(0.2) \approx \frac{1}{h}\left(\Delta f(0.2) - \frac{\Delta^2 f(0.2)}{2} + \frac{\Delta^3 f(0.2)}{3} - \frac{\Delta^4 f(0.2)}{4}\right) = \frac{1}{0.2}\left(2.4 - \frac{8}{2} + \frac{9.6}{3} - \frac{3.84}{4}\right) = 3.2.$

$f''(0) \approx \frac{1}{h^2}\left(\Delta^2 f(0) - \Delta^3 f(0) + \frac{11}{12}\Delta^4 f(0) - \frac{5}{6}\Delta^5 f(0)\right) = \frac{1}{(0.2)^2}\left(2.24 - 5.76 + \frac{11}{12} \times 3.84 - \frac{5}{6} \times 0\right) \approx 0.$ Thus, we have $f''(0) \approx 0$ and $f'(0.2) \approx 3.2$.

Example:

Find $y'(2.2)$ and $y''(2.2)$ from the table:

x	1.4	1.6	1.8	2.0	2.2
$y = y(x)$	4.0552	4.9530	6.0496	7.3891	9.0250

Solution: Since $x = 2.2$ occurs at the end of the table, it is appropriate to use backward difference formulae for derivatives. The backward difference table is

x	$y(x)$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1.4	4.0552				
		0.8978			
1.6	4.9530		0.1988		
		1.0966		0.0441	
1.8	6.0496		0.2429		0.0094
		1.3395		0.0535	
2.0	7.3891		0.2964		
		1.6359			
2.2	9.0250				

Here $h = 02$.

$y^{'}(2.2) \approx \frac{1}{h}\left(\nabla y(2.2) + \frac{\nabla^2 y(2.2)}{2} + \frac{\nabla^3 y(2.2)}{3} + \frac{\nabla^4 y(2.2)}{4}\right) = \frac{1}{0.2}\left(1.6359 + \frac{0.2964}{2} + \frac{0.0535}{3} + \frac{0.0094}{4}\right) = 9.0215.$

$y^{''}(2.2) \approx \frac{1}{h^2}\left(\nabla^2 y(2.2) + \nabla^3 y(2.2) + \frac{11}{12}\nabla^4 y(2.2)\right) = \frac{1}{(2.2)^2}\left(0.2964 + 0.0535 + \frac{11}{12} \times 0.0094\right) = 8.9629$. Thus, we have $y^{'}(2.2) \approx 9.0215$ and $y^{''}(2.2) \approx 8.9629$.

Remark:

The form of a central difference table is:

$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{\delta y}$	$\boldsymbol{\delta^2 y}$	$\boldsymbol{\delta^3 y}$	$\boldsymbol{\delta^4 y}$	$\cdots$	$\cdots$
x_0	y_0						
		$\delta y_{1/2}$					
x_1	y_1		$\delta^2 y_1$				
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$			
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$	$\cdots$	$\cdots$
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$			
x_3	y_3		$\delta^2 y_3$		$\vdots$		
		$\delta y_{7/2}$		$\vdots$			
x_4	y_4		$\vdots$		$\vdots$		
		$\vdots$		$\vdots$			
$\vdots$	$\vdots$		$\vdots$		$\vdots$		

Example:

From the following table of values, estimate $y'(2)$ and $y''(2)$.

x	0	1	2	3	4
y	6.9897	7.4036	7.7815	8.1281	8.4510

Solution: First, let us take $h = 2$. Then

$y'(2) \approx \frac{1}{h}\delta y(2) = \frac{1}{2}(y(2+1) - y(2-1)) = \frac{y(3)-y(1)}{2} = \frac{8.1281-7.4036}{2} =$

$\frac{0.7245}{2} = 0.36225.$

Thus, $y'(2) \approx 0.36225.$

To evaluate $y''(2)$, let us take $h = 1$. Then, we have the following central difference table.

x	y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
0	6.9897				
		0.4139			
1	7.4036		-0.0360		
		0.3779		0.0047	
2	7.7815		-0.0313		0.0029
		0.3466		0.0076	
3	8.1281		-0.0237		
		0.3229			
4	8.4510				

$y''(2) \approx \frac{1}{h^2}\left(\delta^2 y(2) - \frac{1}{12}\delta^4 y(2)\right) = \frac{1}{1^2}\left(-0.0313 - \frac{1}{12} \times 0.0029\right) = -0.0315.$

Thus, $y''(2) \approx -0.0315$ and $y'(2) \approx 0.36225.$

Remark:

There is another formula:

$$(7)\, f'(x) = \frac{1}{h}\left(\mu\delta f(x) - \frac{1}{6}\mu\delta^3 f(x) + \frac{1}{30}\mu\delta^5 f(x) - \cdots \ldots\right),$$

where

$$\mu\delta f(x) = \frac{\delta f\left(x+\frac{h}{2}\right)+\delta f\left(x-\frac{h}{2}\right)}{2},$$

$$\mu\delta^3 f(x) = \frac{\delta^3 f\left(x+\frac{h}{2}\right)+\delta^3 f\left(x-\frac{h}{2}\right)}{2},$$

$$\mu\delta^5 f(x) = \frac{\delta^5 f\left(x+\frac{h}{2}\right)+\delta^5 f\left(x-\frac{h}{2}\right)}{2},$$

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Let us apply this formula to the previous example to evaluate $y'(2)$. Take $h = 1$, and consider the central difference table constructed in the solution of the previous example.

$$\mu\delta y(x) = \frac{\delta y\left(\frac{5}{2}\right)+\delta y\left(\frac{3}{2}\right)}{2} = \frac{0.3466+0.3779}{2} = 0.36225.$$

$$\mu\delta^3 y(x) = \frac{\delta^3 y\left(\frac{5}{2}\right)+\delta^3 y\left(\frac{3}{2}\right)}{2} = \frac{0.0076+0.0047}{2} = 0.00615.$$

Therefore, we have

$$y'(2) \approx \frac{1}{h}\left(\mu\delta y(2) - \frac{1}{6}\mu\delta^3 y(2)\right) = \frac{1}{1}\left(0.36225 - \frac{1}{6}\times 0.00615\right) = 0.361225.$$

Thus, we have $y'(2) \approx 0.361225$.

Thus, we may use the formula (7) instead of (5) to find first order derivatives. We may use the formula (6) to find second order derivatives.

We had described methods to find derivatives at node points $x_0 + ih$. However, to find derivatives at non-node points, we need the next technique based on interpolating polynomials. The next technique is applicable also for node points.

Numerical differentiation based on interpolating polynomials:

Suppose that the values of a function $f(x)$ are given at the $(n+1)$ distinct nodes $x_0, x_1, \dots, x_n$. Suppose $f(x_i) = f_i = y_i$. Let $p_n(x)$ be any (the) polynomial of degree $\leq n$, which interpolates $f(x)$ at the nodes $x_0, x_1, \dots, x_n$. That is, $p_n(x_i) = f(x_i)$, for $i = 0, 1, 2, \dots, n$. The polynomial $p_n(x)$ may be taken in Lagrange's form or in Newton's divided difference form (or in Newton's forward (or) backward difference form, if points are equispaced). Then we have a fundamental assumption: $f(x) \approx p_n(x)$, to have interpolations. We extend this idea to find derivatives. We assume that $f'(x) \approx p_n'(x), f''(x) \approx p_n''(x), f'''(x) \approx p_n'''(x), \dots$, to find derivatives.

Example:

Find $f'(0.25)$ and $f'(0.22)$ from the following data by using an interpolating polynomial.

x	0.15	0.21	0.23	0.27
$y = f(x)$	0.1761	0.3222	0.3617	0.4314

Solution: We first construct the divided difference table for the given data.

x	$f[\,]$	$f[\,,\,]$	$f[\,,\,,\,]$	$f[\,,\,,\,,\,]$
0.15	0.1761			
		2.4350		
0.21	0.3222		-5.7500	
		1.9750		15.6250
0.23	0.3617		-3.8750	
		1.7425		
0.27	0.4314			

Let us take $x_0 = 0.15$, $x_1 = 0.21$, $x_2 = 0.23$, $x_3 = 0.27$. Then, $f[x_0] = 0.1761$, $f[x_0, x_1] = 2.4350$, $f[x_0, x_1, x_2] = -5.7500$, $f[x_0, x_1, x_2, x_3] = 15.6250$.

Therefore, the Newton divided difference polynomial is

$p_3(x) = f[x_0] + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)f[x_0, x_1, x_2, x_3]$.

Then

$f'(x) \approx p_3'(x) = f[x_0, x_1] + \{(x - x_0) + (x - x_1)\}f[x_0, x_1, x_2] + \{(x - x_1)(x - x_3) + (x - x_0)(x - x_3) + (x - x_0)(x - x_2)\}f[x_0, x_1, x_2, x_3]$.

Therefore,

$f'(0.25) \approx p_3'(0.25) = 2.435 + \{(0.25 - 0.15) + (0.25 - 0.21)\} \times (-5.75) + \{(0.25 - 0.21)(0.25 - 0.23) + (0.25 - 0.15)(0.25 - 0.23) + (0.25 - 0.15)(0.25 - 0.21)\} \times 15.625 = 1.7363$.

Also,

$f'(0.22) \approx p_3'(0.22) = 2.435 + \{(0.22 - 0.15) + (0.22 - 0.21)\} \times (-5.75) + \{(0.22 - 0.21)(0.22 - 0.23) + (0.22 - 0.15)(0.22 - 0.23) + (0.22 - 0.15)(0.22 - 0.21)\} \times 15.625 = 1.9734.$

Thus, we have $f'(0.25) \approx 1.7363$ and $f'(0.22) \approx 1.9374$.

Example:

Use the Lagrange's interpolation polynomial fitting the points $y(1) = -3$, $y(3) = 0$, $y(4) = 30$, $y(6) = 132$ to find $y'(5)$ and $y''(5)$.

Solution: The given data can be arranged as follows.

x	1	3	4	6
$y = y(x)$	-3	0	30	132

The Lagrange's interpolation polynomial is

$p_3(x) = \frac{(x-3)(x-4)(x-6)}{(1-3)(1-4)(1-6)} \times (-3) + \frac{(x-1)(x-4)(x-6)}{(3-1)(3-4)(3-6)} \times 0 + \frac{(x-1)(x-3)(x-6)}{(4-1)(4-3)(4-6)} \times 30 + \frac{(x-1)(x-3)(x-4)}{(6-1)(6-3)(6-4)} \times 132 = \frac{1}{2}(-x^3 + 27x^2 - 92x + 60).$

$y'(x) \approx p_3'(x) = \frac{1}{2}(-3x^2 + 54x - 92)$. Therefore, $y'(5) \approx p_3'(5) = \frac{1}{2}(-3 \times 5^2 + 54 \times 5 - 92) = 54$. $y''(x) \approx p_3''(x) = \frac{1}{2}(-6x + 54)$. Therefore, $y''(5) \approx p_3''(5) = \frac{1}{2}(-6 \times 5 + 54) = 12$. Thus, we have $y'(5) \approx 54$ and $y''(5) \approx 12$.

Remark:

One may find that Newton's polynomial has more computational advantages than Lagrange's polynomial.

Specific differentiation formulae:

Suppose that $x_i = x_0 + ih$ for a step length $h > 0$. Assume that the function values $f(x_i) = f_i = y_i$ are given at these equispaced points x_i. The relations (1), (2), (3), (4), (5), (6), and (7) given in the beginning of this chapter give the following relations:

$$f'(x_0) \approx \frac{1}{h}\Delta f(x_0) = \frac{f(x_1)-f(x_0)}{h} = \frac{f(x_1)-f(x_0)}{x_1-x_0}.$$

$$f''(x_0) \approx \frac{1}{h^2}\Delta^2 f(x_0) = \frac{f(x_2)-2f(x_1)+f(x_0)}{h^2} = \frac{f(x_2)-2f(x_1)+f(x_0)}{(x_2-x_1)(x_1-x_0)}.$$

$$f'(x_n) \approx \frac{1}{h}\nabla f(x_n) = \frac{f(x_n)-f(x_{n-1})}{h} = \frac{f(x_n)-f(x_{n-1})}{x_n-x_{n-1}}.$$

$$f''(x_n) \approx \frac{1}{h^2}\nabla^2 f(x_n) = \frac{f(x_n)-2f(x_{n-1})+f(x_{n-2})}{h^2} = \frac{f(x_n)-2f(x_{n-1})+f(x_{n-2})}{(x_n-x_{n-1})(x_{n-1}-x_{n-2})}.$$

$f'(x) \approx \frac{1}{h}\delta f(x) = \frac{f\left(x+\frac{h}{2}\right)-f\left(x-\frac{h}{2}\right)}{h}$ (or; $f'(x) \approx \frac{f(x+h)-f(x-h)}{2h}$ on replacing h by $2h$).

$$f''(x) \approx \frac{1}{h^2}\delta^2 f(x) = \frac{f(x+h)-2f(x)+f(x-h)}{h^2}.$$

Such specific formulae may be applied to find numerical derivatives at node points. We shall use such specific formulae to solve differential equations in Chapter 6.

Example:

Evaluate $\cos(0.19)$ by using the formula $f'(x) \approx \frac{f(x+h)-f(x-h)}{2h}$ and the following table.

x	0.17	0.19	0.21
$\sin x$	0.16918	0.18886	0.20846

Solution: Here $f(x) = \sin x$ and $f'(x) = \cos x$. Take $h = 0.02$. Then,

$\cos(0.19) = f'(0.19) \approx \frac{f(0.21)-f(0.17)}{2\times 0.02} = \frac{0.20846-0.16918}{0.04} = \frac{0.03928}{0.04} = 0.982.$

Thus, we have $\cos(0.19) \approx 0.982.$

Example:

Evaluate $y''(0.3)$ by using the formula $y''(x) \approx \frac{y(x+2h)-2y(x+h)+y(x)}{h^2}$ and the following table.

x	0.1	0.2	0.3	0.4	0.5	0.6
$y(x)$	2.631	3.328	4.097	4.944	5.875	6.896

Solution: Take $h = 0.1$. Then,

$y''(0.3) \approx \frac{y(0.5)-2y(0.4)+y(0.3)}{(0.1)^2} = \frac{5.875-2\times 4.944+4.097}{0.01} = 8.4$. Thus, we have

$y''(0.3) \approx 8.4.$

Final Remarks:

These specific differentiation formulae mentioned above clearly reveal that we can design our own formulae for numerical differentiation such that in the limiting cases we do have exact value for differentiation. However, it should be mentioned that all formulae mentioned in this chapter are classical formulae, and they had been designed in view of deriving error analysis for these formulae.

Exercises:

(1) Find the first derivative of $f(x)$ at $x = 0.4$ from the following table:

x	0.1	0.2	0.3	0.4
$f(x)$	1.10517	1.22140	1.34986	1.49182

(2) Find the second derivative of $f(x)$ at $x = 1.3$ from the following table:

x	1.3	1.5	1.7	1.9	2.1	2.4
$f(x)$	2.9648	2.6599	2.3333	1.9922	1.6442	1.2969

(3) A slider in a machine moves along a fixed straight rod. Its distance s cm along the rod is given below for various values of time t seconds. Find the velocity of the slider and its acceleration when $t = 0.3$ seconds.

t	0.0	0.1	0.2	0.3	0.4	0.5	0.6

s	3.013	3.162	3.287	3.364	3.395	3.381	3.324

(Hint: $velocity = \frac{ds}{dt}$, $acceleration = \frac{d^2s}{dt^2}$. Use central difference formulae.)

(4) Find $y'(2.5)$ and $y''(-1.5)$ from the following data:

x	-2	-1	2	3
$y(x)$	-12	-8	3	5

(Hint: Use an interpolating polynomial.)

(5) Given the table of values

x	150	152	154	156
$\sqrt{x}$	12.247	12.329	12.410	12.490

evaluate $\frac{1}{2\sqrt{152}}$ by using the formula $f'(x) \approx \frac{f(x+h)-f(x-h)}{2h}$. (Hint: $\frac{d\sqrt{x}}{dx} = \frac{1}{2\sqrt{x}}$)

CHAPTER- 5

Numerical Integration

Let us recall the definition of the Riemann integration. The Riemann integral $\int_a^b f(x)dx$ is the limit of the sums

(1) $\sum_{i=0}^{N-1} f(\alpha_i)(x_{i+1} - x_i)$

where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, and $\alpha_i \in [x_i, x_{i+1}]$; as $\max_i(x_{i+1} - x_i) \to 0$. Write $h = \frac{b-a}{N}$, and $(x_{i+1} - x_i) = w_i h$. Then (1) can be written as

$$\sum_{i=0}^{N-1} f(\alpha_i) w_i h = (b-a) \sum_{i=0}^{N-1} \left(\frac{w_i}{N}\right) f(\alpha_i)$$

where

$$\sum_{i=0}^{N-1} \frac{w_i}{N} = \sum_{i=0}^{N-1} \frac{(x_{i+1} - x_i)}{hN} = \frac{1}{hN} \times (b-a) = 1.$$

Therefore, if we write $\lambda_i = \frac{w_i}{N}$, then (1) is further reduced to the form $(b - a) \sum_{i=0}^{N-1} \lambda_i f(\alpha_i)$, where $\sum_{i=0}^{N-1} \lambda_i = 1$. Thus, we have

(2) $\int_a^b f(x)dx \approx (b-a) \sum_{i=0}^{N-1} \lambda_i f(\alpha_i)$

where $0 < \lambda_i < 1$ and $\sum_{i=0}^{N-1} \lambda_i = 1$.

Some formulae of the type (2) will be used to evaluate an approximate value for integration. In this chapter, five numerical integration rules, or, formulae (or; numerical quadrature rule, or, formulae) are presented. All of them are of the type (2). They are: (i) Trapezoidal rule, (ii) Simpson's (1/3)-rd rule, (iii) Simpson's (3/8)-th rule, (iv) Boole's rule, and (v) Weddle's rule.

We omit the derivations of these rules. For example, if $p_n(x)$ is an interpolating polynomial of $f(x)$, then we may assume that $\int_a^b f(x)dx \approx \int_a^b p_n(x)dx$ to derive a formula. If an integration rule is of type (2), we may derive λ_i under the assumption that the rule (2) is exact for polynomials of degree $\leq n$. There are some more derivation methods to derive formulae for numerical integrations.

Boole's formula:

$$\int_a^b f(x)dx \approx \frac{2h}{45}\{7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)\},$$

where $x_0 = a$, $x_4 = b$, $h = \frac{b-a}{4}$, $x_1 = a + h = x_0 + h$, $x_2 = x_0 + 2h$, and $x_3 = x_0 + 3h$.

Weddle's formula:

$$\int_a^b f(x)dx \approx \frac{3h}{10}\{f(x_0) + 5f(x_1) + f(x_2) + 6f(x_3) + f(x_4) + 5f(x_5) + f(x_6)\},$$

where $x_0 = a$, $x_6 = b$, $h = \frac{b-a}{6}$, and $x_i = x_0 + ih$ for $i = 0, 1, 2, 3, 4, 5, 6$.

Trapezoidal rule:

Write $h = \frac{b-a}{N}$, where N is a given positive integer. Write $x_0 = a, x_N = b$, and $x_i = x_0 + ih$ for $i = 0, 1, 2, \dots, N$. Then the trapezoidal rule with N subintervals or $(N + 1)$ node points is

$$(1) \int_a^b f(x)dx \approx \frac{h}{2}[f(x_0) + 2\{f(x_1) + f(x_2) + \cdots + f(x_{N-1})\} + f(x_N)].$$

If we take $N = 1$, then this formula reduces to

$$(2) \int_a^b f(x)dx \approx (b-a) \times \frac{f(a)+f(b)}{2},$$

which represents the area of the trapezium formed by x-axis, the ordinates $x = a, x = b$, and the straight line joining the points $(a, f(a))$ and $(b, f(b))$. The formula (2) is said to be in simple form and the formula (1) is said to be in composite form. The formula (1) can be derived by applying (2) to each $\int_{x_{i-1}}^{x_i} f(x)dx$ and by finding the sum $\sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} f(x)dx = \int_a^b f(x)\, dx$.

Simpson's (1/3)rd rule:

Write $h = \frac{b-a}{2N}$, where N is a given positive integer. Write $x_0 = a$, $x_{2N} = b$, and $x_i = x_0 + ih$ for $i = 0, 1, 2, \dots, 2N$. Then the Simpson's (1/3)rd rule with $2N$ subintervals or $(2N + 1)$ node points is

$$(3) \int_a^b f(x)dx \approx \frac{h}{3}[f(x_0) + 4\{f(x_1) + f(x_3) + f(x_5) + \cdots + f(x_{2N-1})\} +$$

$$2\{f(x_2) + f(x_4) + \cdots + f(x_{2N-2})\} + f(x_{2N})].$$

If we take $N = 1$, then this formula reduces to

$$(4) \int_a^b f(x)dx \approx \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right].$$

The formula (4) is said to be in simple form and the formula (3) is said to be in composite form. The formula (3) can be derived by applying (4) to each $\int_{x_{i-1}}^{x_i} f(x)dx$ and by finding their sum.

Simpson's (3/8)th rule:

Write $h = \frac{b-a}{3N}$, where N is a given positive integer. Write $x_0 = a$, $x_{3N} = b$, and $x_i = x_0 + ih$ for $i = 0, 1, 2, \dots, 3N$. Then the Simpson's (3/8)th rule with $3N$ subintervals or $(3N + 1)$ node points is

$$(5) \int_a^b f(x)dx \approx \frac{3h}{8}\left[f(x_0) + 3\{(f(x_1) + f(x_2)) + (f(x_4) + f(x_5)) + (f(x_7) + f(x_8)) + \cdots + (f(x_{3N-2}) + f(x_{3N-1})\} + 2\{f(x_3) + f(x_6) + f(x_9) + \cdots + f(x_{3N-3})\} + f(x_{3N})\right].$$

If we take $N = 1$, then this formula reduces to

$$(6) \int_a^b f(x)dx \approx \frac{(b-a)}{8}\left[f(a) + 3f\left(a + \frac{b-a}{3}\right) + 3f\left(a + 2 \times \frac{b-a}{3}\right) + f(b)\right].$$

The formula (6) is said to be in simple form and the formula (5) is said to be in composite form. The formula (5) can be derived by applying (6) to each $\int_{x_{i-1}}^{x_i} f(x)dx$ and by finding their sum.

Remark:

It is possible to derive composite forms for Boole's rule and Weddle's rule. However, we apply them only in their simple forms, even though we also apply the trapezoidal rule and Simpson's rule by their composite forms. Note that it would be sufficient for all calculations in terms of these rules, if a function $f(x)$ is given only at the node points.

Example:

Evaluate $\int_0^1 \frac{1}{1+x}\, dx$ by using Boole's rule.

Solution: The Boole's rule is

$$\int_a^b f(x)dx \approx \frac{2h}{45}\{7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)\},$$

where $x_0 = a, x_4 = b$, and $x_i = x_0 + ih$. For the given problem, we have $a = 0, b = 1, h = \frac{1}{4}, x_0 = 0, x_1 = \frac{1}{4}, x_2 = \frac{1}{2}, x_3 = \frac{3}{4}$, and $x_4 = 1$. Therefore, we have

$$\int_0^1 \frac{1}{1+x}\, dx \approx \frac{2}{45} \times \frac{1}{4} \times \left(\frac{7}{1+0} + \frac{32}{1+\frac{1}{4}} + \frac{12}{1+\frac{1}{2}} + \frac{32}{1+\frac{3}{4}} + \frac{7}{1+1}\right) = \frac{1}{90} \times (7 + 25.6 + 8 + 18.2857 + 3.5) = \frac{62.3857}{90} = 0.6932.$$

Thus, we have $\int_0^1 \frac{1}{1+x}\, dx \approx 0.6932$.

Example:

Apply Boole's rule to the interval [0, ½] and to the interval [1/2, 1], separately, and hence evaluate $\int_0^1 \frac{dx}{1+x}$.

Solution: The Boole's rule is

$\int_a^b f(x)dx \approx \frac{2h}{45}\{7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)\}$,

where $x_0 = a, x_4 = b$, and $x_i = x_0 + ih$.

Write $I_1 = \int_0^{1/2} \frac{dx}{1+x}$ and $I_2 = \int_{1/2}^1 \frac{dx}{1+x}$. By Boole's rule, we have

$I_1 = \int_0^{1/2} \frac{dx}{1+x} \approx \frac{2}{45} \times \frac{\frac{1}{2}-0}{4} \times \left(\frac{7}{1+0} + \frac{32}{1+\frac{1}{8}} + \frac{12}{1+\frac{2}{8}} + \frac{32}{1+\frac{3}{8}} + \frac{7}{1+\frac{1}{2}}\right) = \frac{1}{180} \times (7 +$

$28.4444 + 9.6 + 23.2727 + 4.6667) = \frac{72.9838}{180} = 0.4055.$

$I_2 = \int_{1/2}^1 \frac{dx}{1+x} \approx \frac{2}{45} \times \frac{1-\frac{1}{2}}{4} \times \left(\frac{7}{1+\frac{1}{2}} + \frac{32}{1+\frac{5}{8}} + \frac{12}{1+\frac{6}{8}} + \frac{32}{1+\frac{7}{8}} + \frac{7}{1+1}\right) = \frac{1}{180} \times (4.6667 +$

$19.6923 + 6.8571 + 17.0667) = \frac{48.2828}{180} = 0.2682.$

Therefore, $\int_0^1 \frac{dx}{1+x} = I_1 + I_2 \approx 0.4055 + 0.2682 = 0.6737$. Thus, we have $\int_0^1 \frac{dx}{1+x} \approx 0.6737.$

Example:

Evaluate $\int_{1.2}^{1.6} f(x)dx$ from the following table, by using Boole's rule.

x	1.2	1.3	1.4	1.5	1.6

$f(x)$	0.9320	0.9636	0.9855	0.9975	0.9996

Solution: The Boole's rule is

$\int_a^b f(x)dx \approx \frac{2h}{45}\{7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)\}.$

Here $x_0 = a = 1.2,\ x_1 = 1.3,\ x_2 = 1.4,\ x_3 = 1.5,\ x_4 = b = 1.6,$ and $h = 0.1.$

Therefore, we have

$\int_{1.2}^{1.6} f(x)\,dx \approx \frac{2\times 0.1}{45} \times (7 \times 0.9320 + 32 \times 0.9636 + 12 \times 0.9855 + 32 \times$

$0.9975 + 7 \times 0.9996) = \frac{0.2}{45} \times 87.8324 \approx 0.3904.$

Thus, we have $\int_{1.2}^{1.6} f(x)dx \approx 0.3904.$

Example:

Evaluate $\int_0^{0.4} f(x)dx$ and $\int_{0.4}^{0.8} f(x)dx$ by using Boole's rule from the following table.

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
$f(x)$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0

Hence find $\int_0^{0.8} f(x)dx$.

Solution: The Boole's rule is

$\int_a^b f(x)dx \approx \frac{2h}{45}\{7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)\},$

where $x_0 = a, x_4 = b,$ and $x_i = x_0 + ih.$

$\int_0^{0.4} f(x)dx \approx \frac{2\times0.1}{45} \times (7 \times 0.2 + 32 \times 0.3 + 12 \times 0.4 + 32 \times 0.5 + 7 \times 0.6) =$

$\frac{0.2}{45} \times 36.2 = 0.1609.$

$\int_{0.4}^{0.8} f(x)dx \approx \frac{2\times0.1}{45} \times (7 \times 0.6 + 32 \times 0.7 + 12 \times 0.8 + 32 \times 0.9 + 7 \times 1.0) =$

$\frac{0.2}{45} \times 72 = 0.32.$

Therefore, $\int_0^{0.8} f(x)dx \approx 0.1609 + 0.32 = 0.4809$. Thus, we have $\int_0^{0.8} f(x)dx \approx 0.4809.$

Example:

Apply Weddle's rule to evaluate $\int_0^1 \frac{dx}{1+x}$.

Solution: Weddle's rule is

$\int_a^b f(x)dx \approx \frac{3h}{10}\{f(x_0) + 5f(x_1) + f(x_2) + 6f(x_3) + f(x_4) + 5f(x_5) + f(x_6)\},$

where $x_0 = a$, $x_6 = b$, $h = \frac{b-a}{6}$, and $x_i = x_0 + ih$. Here $f(x) = \frac{1}{1+x}$, $a = x_0 = 0$, and $h = \frac{1-0}{6} = \frac{1}{6}$. Therefore, we have

$\int_0^1 \frac{dx}{1+x} \approx \frac{3}{10} \times \frac{1}{6} \times \left(\frac{1}{1+0} + \frac{5}{1+\frac{1}{6}} + \frac{1}{1+\frac{2}{6}} + \frac{6}{1+\frac{3}{6}} + \frac{1}{1+\frac{4}{6}} + \frac{5}{1+\frac{5}{6}} + \frac{1}{1+1}\right) = \frac{1}{20} \times$

$(1 + 4.2857 + 0.75 + 4 + 0.6 + 2.7273 + 0.5) = \frac{13.863}{20} = 0.69375.$

Thus, we have $\int_0^1 \frac{dx}{1+x} \approx 0.69375.$

Example:

Evaluate $\int_0^{0.6} f(x)dx$ from the table

x	0	0.1	0.2	0.3	0.4	0.5	0.6
$f(x)$	0.4	0.5	0.6	0.7	0.8	0.9	1.0

by using the Weddle's rule.

Solution: Weddle's rule is

$$\int_a^b f(x)dx \approx \frac{3h}{10}\{f(x_0) + 5f(x_1) + f(x_2) + 6f(x_3) + f(x_4) + 5f(x_5) + f(x_6)\},$$

where $x_0 = a,\ \ x_6 = b,\ h = \frac{b-a}{6}$, and $x_i = x_0 + ih.$ Here $f(x) = \frac{1}{1+x}$, $a = x_0 = 0, b = x_6 = 0.6,$ and $h = 0.1.$ Therefore, we have

$$\int_0^{0.6} f(x)dx \approx \frac{3\times0.1}{10} \times (0.4 + 5 \times 0.5 + 0.6 + 6 \times 0.7 + 0.8 + 5 \times 0.9 + 1) =$$

$$0.03 \times 14 = 0.42.$$

Thus, we have $\int_0^{0.6} f(x)dx \approx 0.42.$

Example:

Find approximate values of $\int_0^{\pi} \sin x\, dx$, by using (i) Trapezoidal rule, (ii) Simpson's 1/3 rule (or simply: Simpson's rule), (iii) Simpson's 3/8 rule, by dividing the range of integration into 6 subintervals (7 point formulae).

Solution: (i) The trapezoidal rule with 6 subintervals is

$$\int_a^b f(x)\, dx \approx \frac{h}{2}[f(x_0) + 2\{f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)\} + f(x_6)]$$

where $x_0 = a$, $x_6 = b$, $h = \frac{b-a}{6}$, and $x_i = x_0 + ih$. Here, $f(x) = \sin x$, $a = x_0 = 0$, $b = x_6 = \pi$, and $h = \frac{\pi - 0}{6} = \frac{\pi}{6}$. We take $\pi \approx 3.1415$. Therefore,

$\int_0^\pi \sin x \, dx \approx \frac{\pi}{6\times2} \times \left[\sin 0 + 2 \times \left\{\sin\frac{\pi}{6} + \sin\frac{2\pi}{6} + \sin\frac{3\pi}{6} + \sin\frac{4\pi}{6} + \sin\frac{5\pi}{6}\right\} + \sin\pi\right] = \frac{3.1415}{12} \times [0 + 2 \times \{0.5 + 0.8660 + 1 + 0.8660 + 0.5\} + 0] \approx 1.9540$.

Thus, we have $\int_0^\pi \sin x \, dx \approx 1.9540$.

(ii) The Simpson's 1/3 rule with 6 subintervals is

$$\int_a^b f(x)dx \approx \frac{h}{3}[f(x_0) + 4\{f(x_1) + f(x_3) + f(x_5)\} + 2\{f(x_2) + f(x_4)\} + f(x_6)]$$

where $x_0 = a$, $x_6 = b$, $h = \frac{b-a}{6}$, and $x_i = x_0 + ih$. Here, $f(x) = \sin x$, $a = x_0 = 0$, $b = x_6 = \pi$, and $h = \frac{\pi - 0}{6} = \frac{\pi}{6}$. We take $\pi \approx 3.1415$. Therefore,

$\int_0^\pi \sin x \, dx \approx \frac{\pi}{6\times3} \times \left[\sin 0 + 4 \times \left\{\sin\frac{\pi}{6} + \sin\frac{3\pi}{6} + \sin\frac{5\pi}{6}\right\} + 2 \times \left\{\sin\frac{2\pi}{6} + \sin\frac{4\pi}{6}\right\} + \sin\pi\right] = \frac{3.1415}{18} \times [0 + 4 \times \{0.5 + 1 + 0.5\} + 2 \times \{0.8660 + 0.8660\} + 0] \approx 2.0008$. Thus, we have $\int_0^\pi \sin x \, dx \approx 2.0008$.

(iii) The Simpson's 3/8 rule with 6 subintervals is

$$\int_a^b f(x)dx \approx \frac{3h}{8}[f(x_0) + 3\{f(x_1) + f(x_2) + f(x_4) + f(x_5)\} + 2\{f(x_3)\} + f(x_6)]$$

where $x_0 = a$, $x_6 = b$, $h = \frac{b-a}{6}$, and $x_i = x_0 + ih$. Here, $f(x) = \sin x$, $a = x_0 = 0$, $b = x_6 = \pi$, and $h = \frac{\pi - 0}{6} = \frac{\pi}{6}$. We take $\pi \approx 3.1415$. Therefore, $\int_0^{\pi} \sin x \; dx \approx \frac{3\pi}{8 \times 6} \times \left[\sin 0 + 3 \times \left\{\sin\frac{\pi}{6} + \sin\frac{2\pi}{6} + \sin\frac{4\pi}{6} + \sin\frac{5\pi}{6}\right\} + 2 \times \left\{\sin\frac{3\pi}{6}\right\} + \sin\pi\right] = \frac{3.1415}{16} \times [0 + 3 \times \{0.5 + 0.8660 + 0.8660 + 0.5\} + 2 \times \{1\} + 0] = \frac{3.1415}{16} \times 10.196 = 2.0019$. Thus, we have $\int_0^{\pi} \sin x \, dx \approx 2.0019$.

Example:

From the following data, estimate the value of $\int_1^5 \log x \; dx$ by using Simpson's 1/3 rule (or; Simpson's rule).

x	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
$f(x)$	0	0.4055	0.6931	0.9162	1.0986	1.2528	1.3863	1.5041	1.6094

Solution: For the given problem $a = x_0 = 1$, $b = x_8 = 5$, and $h = 0.5$. The corresponding 8 subintervals Simpson's 1/3 rule for the given problem becomes $\int_1^5 \log x \; dx \approx \frac{0.5}{3} \times [\log 1 + 4 \times \{\log 1.5 + \log 2.5 + \log 3.5 + \log 4.5\} + 2 \times \{\log 2 + \log 3 + \log 4\} + \log 5] = \frac{0.5}{3} \times [0 + 4 \times \{0.4055 + 0.9163 + 1.2528 + 1.5041\} + 2 \times \{0.6931 + 1.0986 + 1.3863\} + 1.6094] \approx 4.0467$.

Thus, we have $\int_1^5 \log x \; dx \approx 4.0467$.

Example:

Evaluate $\int_0^1 f(x)dx$ by using the trapezoidal rule, where values of $f(x)$ are given in the following table.

x	0	1/4	1/2	3/4	1
$f(x)$	1	0.9412	0.8	0.64	0.5

Solution: Here $a = x_0 = 0$, $b = x_4 = 1$, and $h = 1/4$. The 4-subintervals trapezoidal rule gives

$\int_0^1 f(x)dx \approx \frac{1}{4\times2} \times \left[f(0) + 2\left\{f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right)\right\} + f(1)\right] = \frac{1}{8} \times$ $[1 + 2 \times \{0.9412 + 0.8 + 0.64\} + 0.5] = 0.7828$. Thus, we have $\int_0^1 f(x)dx \approx$ 0.7828.

Example:

Evaluate $\int_0^{0.9} f(x)dx$ from the table

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$f(x)$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0

by using Simpson's (3/8)th rule.

Solution: Here $a = x_0 = 0$, $b = x_9 = 0.9$, and $h = 0.1$. The 9 subintervals Simpson's (3/8)th rule gives

$\int_0^{0.9} f(x)dx \approx \frac{3}{8} \times 0.1 \times [0.1 + 3 \times \{0.2 + 0.3 + 0.5 + 0.6 + 0.8 + 0.9\} + 2 \times \{0.4 + 0.7\} + 1.0] = \frac{0.3}{8} \times [0. + 8.4 + 2.2 + 1.0] = 0.43875$. Thus, we have $\int_0^{0.9} f(x)dx \approx 0.43875.$

Example:

Find the area bounded by the curve $y = f(x)$, x-axis, and the ordinates $x = 7.47$ and $x = 7.52$, where $f(x)$ is given in the following table.

x	7.47	7.48	7.49	7.50	7.51	7.52
$f(x)$	1.93	1.95	1.98	2.01	2.03	2.06

Use the trapezoidal rule.

Solution: The required area is $\int_{7.47}^{7.52} f(x)dx$. The 6 point trapezoidal rule gives:

$\int_{7.47}^{7.52} f(x)dx \approx \frac{h}{2}[f(7.47) + 2\{f(7.48) + f(7.49) + f(7.50) + f(7.51)\} + f(7.52)] = \frac{0.01}{2}[1.93 + 2 \times \{1.95 + 1.98 + 2.01 + 2.03\} + 2.06] = 0.0996$.

Thus the required area is 0.0996.

Example:

A solid of revolution is formed by rotating a curve $y = f(x)$ about x-axis between the planes $x = 0$ and $x = 1$. Estimate the volume of this portion, where $f(x)$ is given in the following table.

x	0.00	0.25	0.50	0.75	1.00

$y = f(x)$	1.0000	0.9896	0.9589	0.9089	0.8415

Use the Simpson's rule (that is, Simpson's 1/3 rule).

Solution: The required volume is $V = \pi \int_0^1 y^2\, dx = \pi \int_0^1 (f(x))^2\, dx$.

Therefore, by the Simpson's 1/3 rule, we have

$V = \pi \int_0^1 (f(x))^2\, dx \approx \pi \times \frac{h}{3} \times [(f(0))^2 + 2\{(f(0.5))^2\} + 4\{(f(0.25))^2 +$

$(f(0.75))^2\} + (f(1))^2] = \pi \times \frac{0.25}{3} \times [1^2 + 2 \times \{0.9589^2\} + 4 \times \{0.9896^2 +$

$0.9089^2\} + 0.8415^2] \approx 2.8192.$

Thus, the required volume is 2.8192.

Example:

A missile is launched from a ground station. The initial velocity at time $t = 0\, sec$ is $0\, m/sec$. The acceleration a during its first 80 seconds of flight, as recorded, is given in the following table.

$t\ (sec)$	0	10	20	30	40	50	60	70	80
$a\ (\frac{m}{sec^2})$	30	31.63	33.34	35.47	37.75	40.33	43.25	46.69	50.67

Compute the velocity of the missile when $t = 80$ seconds, by using Simpson's 1/3 rule.

Solution: Note that $\frac{dv}{dt} = a$, and hence $v(80) - v(0) = \int_0^{80} \frac{dv}{dt}\, dt = \int_0^{80} a\, dt$.

Therefore, $v(80) = \int_0^{80} a\, dt$, because the initial velocity $v(0) = 0$. Thus, we have to find the required velocity $\int_0^{80} a\, dt$. By Simpson's 1/3 rule,

$\int_0^{80} a\, dt \approx \frac{h}{3}[a(0) + 4\{a(10) + a(30) + a(50) + a(70)\} + 2\{a(20) + a(40) + a(60)\} + a(80)] = \frac{10}{3} \times [30 + 4 \times \{31.63 + 35.47 + 40.33 + 46.79\} + 2 \times \{33.34 + 37.75 + 43.25\} + 50.67] = 3086.1\, m/sec.$

Thus, the required velocity is $3086.1\, m/sec$.

Final Remarks:

We can draw the following conclusions from the examples discussed above. Numerical integrations are applicable when there is no analytic formula for integration for an integrand of integration or when the integrand values are given only at finitely many nodes. One can design numerical integration rule according to the nodes for which integrand values are known for practical purposes; based on the definition of Riemann integration.

Exercises:

(1) Find the area bounded by the curve $y = \frac{1}{1+\frac{1}{x^2}}$, x-axis and the ordinates $x = 0$ and $x = 1$.

(2) Find $\int_{-1}^{+1} e^{-x}\, dx$ by using (i) Boole's rule, (ii) Weddle's rule, (iii) Trapezoidal rule with 8 subintervals, (iv) Simposon,s 1/3 rule with 8 subintervals, and (v) Simpson's 3/8 rule with 9 subintervals.

(3) Evaluate the integral $\int_0^6 (f(x))^2\, dx$ from the following table.

x	0	1	2	3	4	5	6
$f(x)$	1	0	1	4	9	16	25

Use (i) Weddle's rule, (ii) Trapezoidal rule, (iii) Simpson's 1/3 rule, and (iv) Simpson's 3/8 rule.

Also find $\int_0^4 (f(x))^3\, dx$, by using Boole's rule.

(4) Find an approximate value for $\int_0^1 e^{-x^2}\, dx$.

CHAPTER- 6

Numerical Solutions of Ordinary Differential Equations

The differential equation $y' = y$ has the solutions $y = c\,e^x$, where c is any constant. But, the differential equation along with the condition $y(0) = 1$ (That is, the value of y is 1 at $x = 0$) fixes a unique solution $y = e^x$. Numerical solutions are approximate solutions given at finite number of nodes. We should get a fixed solution, when we are about to search for a numerical solution. Without fixing only one solution for searching, we cannot begin a numerical searching process. To ensure this fixedness, we have to impose one condition, and we usually fix a condition called "initial condition" along with the differential equation.

We write a solution of a differential equation as $y = y(x)$ or simply $y(x)$, where x is the independent variable and y is the dependent variable.

The general form of a first order initial value problem is: $\frac{dy}{dx} = f(x, y); y(x_0) = y_0$. The condition $y(x_0) = y_0$ is called the initial condition. This means that the value of y at $x = x_0$ is y_0. We shall discuss several methods to solve an initial value problem.

Euler Method:

Consider an initial value problem: $y' = f(x,y), y(x_0) = y_0$. Let $h > 0$ be a given step length. Write $x_i = x_0 + ih,\ i = 0,1,2,\dots.$ Let $y_i = y(x_i)$ denote the value of the solution function $y(x)$ at $x = x_i$. Let $y_i' = y'(x_i)$ and $f_i = f(x_i, y_i) = f(x_i, y(x_i))$. We have derived a formula in Chapter 4: $y_i' \approx \frac{y_{i+1}-y_i}{h}$. Substitute this in the given equation $y_i' = f(x_i, y_i)$ to get $\frac{y_{i+1}-y_i}{h} \approx f(x_i, y_i)$ or

$$(1)\, y_{i+1} \approx y_i + hf(x_i, y_i).$$

Substitute the given initial condition $y(x_0) = y_0$ in the R.H.S. of (1) to get $y_1 = y(x_1)$. Substitute y_1 in the R.H.S. of (1) to get $y_2 = y(x_2)$. Substitute y_2 in the R.H.S. of (1) to get $y_3 = y(x_3)$. We proceed in this way until we get solution in a required interval $[x_0, b]$ at all the node points x_i inside the interval. The formula (1) is called Euler's formula (method).

Example:

Solve the initial value problem $y' = -y,\ y(0) = 1$ on the interval [0. 0.04] with $h = 0.01$ by using Euler's method.

Solution: Here $f(x,y) = -y, x_0 = 0$, and $y_0 = 1$. The Euler's formula $y_{i+1} \approx y_i + hf(x_i, y_i)$ becomes $y_{i+1} \approx y_i - 0.01\, y_i = 0.99\, y_i$. Then $y(x_1) = y(0.01) = y_1 \approx 0.99\, y_0 = 0.99$, $y(x_2) = y(0.02) = y_2 \approx 0.99\, y_1 = 0.99 \times$

$0.99 = 0.9801$, $y(x_3) = y(0.03) = y_3 \approx 0.99\, y_2 = 0.99 \times 0.9801 = 0.9703$, $y(x_4) = y(0.04) = y_4 \approx 0.99\, y_3 = 0.99 \times 0.9703 = 0.9606.$

Answer: $y(0) = 1$, $y(0.01) \approx 0.99$, $y(0.02) \approx 0.9801$, $y(0.03) \approx 0.9703$, $y(0.04) \approx 0.9606.$

Example:

Solve $\frac{dy}{dx} = \frac{y-x}{y+x}$ on the interval [0, 0.1] subject to: $y = 1$ at $x = 0$. Use Euler's method with $h = 0.02$.

Solution: Here $x_0 = 0$, $y_0 = 1$, and $f(x,y) = \frac{y-x}{y+x}$. The Euler's formula $y_{i+1} \approx y_i + hf(x_i, y_i)$ becomes $y_{i+1} \approx y_i + 0.02 \times \frac{y_i - x_i}{y_i + x_i}$. Therefore,

$y(x_1) = y(0.02) = y_1 \approx y_0 + 0.02 \times \frac{y_0 - x_0}{y_0 + x_0} = 1 + 0.02 \times \frac{1-0}{1+0} = 1.02,$

$y(x_2) = y(0.04) = y_2 \approx y_1 + 0.02 \times \frac{y_1 - x_1}{y_1 + x_1} = 1.02 + 0.02 \times \frac{1.02-0.02}{1.02+0.02} = 1.0392,$

$y(x_3) = y(0.06) = y_3 \approx y_2 + 0.02 \times \frac{y_2 - x_2}{y_2 + x_2} = 1.0392 + 0.02 \times \frac{1.0392-0.04}{1.0392+0.04} = 1.0577,$

$y(x_4) = y(0.08) = y_4 \approx y_3 + 0.02 \times \frac{y_3 - x_3}{y_3 + x_3} = 1.0577 + 0.02 \times \frac{1.0577-0.06}{1.0577+0.06} = 1.0738,$

$y(x_5) = y(0.1) = y_5 \approx y_4 + 0.02 \times \frac{y_4 - x_4}{y_4 + x_4} = 1.0738 + 0.02 \times \frac{1.0738-0.08}{1.0738+0.08} = 1.0910.$

Answer: $y(0) = 1$, $y(0.02) \approx 1.02$, $y(0.04) \approx 1.0392$, $y(0.06) \approx 1.0577$, $y(0.08) \approx 1.0738, y(0.1) \approx 1.0910.$

Remark:

The answer of the previous example suggests that 0 is the initial point and hence $y(0) = 1$ may be considered as initial condition given at the initial point. This gives a reason for the name initial value problem. Boundary value problems have conditions on boundary points of the region in which solutions are considered.

Runge-Kutta methods:

There are many Runge-Kutta methods to solve a first order initial value problem. We present only two Runge-Kutta methods. One is a second order Runge-Kutta method. Another one is a fourth order Runge-Kutta method. The term "order" is the "order of the errors". In general, higher order methods are better than lower order methods. Euler method is a first order method. So, the methods to be presented in this section are better than Euler method. Euler method is also "a" first order Runge-Kutta method (according to general formats of Runge-Kutta methods).

Consider a first order initial value problem: $y' = f(x, y), y(x_0) = y_0$. Consider a step length $h > 0$. Write $x_i = x_0 + ih, y_i = y(x_i)$ and $y_i' = y'(x_i)$. Then "a" second order Runge-Kutta method is

$y_{i+1} = y_i + \frac{1}{2}(k_1 + k_2)$,

where $k_1 = h\,f(x_i, y_i)$

and $k_2 = h\,f(x_i + h, y_i + k_1)$.

"A" fourth order Runge-Kutta method is

$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_2 + k_4)$

where

$k_1 = h\,f(x_i, y_i)$,

$k_2 = h\,f\left(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}\right)$,

$k_3 = h\,f\left(x_i + \frac{h}{2}, y_i + \frac{k_2}{2}\right)$,

$k_4 = h\,f(x_i + h, y_i + k_3)$.

These two methods are called the second order explicit classical Runge-Kutta method and the fourth order explicit classical Runge-Kutta method. Let us now apply these methods to solve problems. However, other specific (Runge-Kutta) methods may also be given and used to solve problems.

Example:

Given $\frac{dy}{dx} = y - x, y(0) = 2$; find the solution in the interval [0.0.4] by using a second order Runge-Kutta method with $h = 0.1$.

Solution: A second order Runge-Kutta method is

$y_{i+1} = y_i + \frac{1}{2}(k_1 + k_2)$,

where $k_1 = h\,f(x_i, y_i)$

and $k_2 = h\,f(x_i + h, y_i + k_1)$.

For the given problem, we have $f(x, y) = y - x$, $x_0 = 0$, $y_0 = 2$ and $h = 0.1$.

Then the method becomes $y_{i+1} = y_i + \frac{1}{2}(k_1 + k_2)$ where $k_1 = 0.1 \times (y_i - x_i)$

and $k_2 = 0.1 \times (y_i + k_1 - x_i - 0.1)$.

To find $y(0.1) = y_1$:

$k_1 = 0.1 \times (y_0 - x_0) = 0.1 \times (2 - 0) = 0.2$,

$k_2 = 0.1 \times (y_0 + k_1 - x_0 - 0.1) = 0.1 \times (2 + 0.2 - 0 - 0.1) = 0.21$,

$y(0.1) = y_1 \approx y_0 + \frac{1}{2}(k_1 + k_2) = 2 + \frac{1}{2}(0.2 + 0.21) = 2.2050$.

To find $y(0.2) = y_2$:

$k_1 = 0.1 \times (y_1 - x_1) = 0.1 \times (2.205 - 0.1) = 0.2105$,

$k_2 = 0.1 \times (y_1 + k_1 - x_1 - 0.1) = 0.1 \times (2.205 + 0.2105 - 0.1 - 0.1) =$

0.22155,

$y(0.2) = y_2 \approx y_1 + \frac{1}{2}(k_1 + k_2) = 2.205 + \frac{1}{2}(0.2105 + 0.22155) \approx 2.4210$.

To find $y(0.3) = y_3$:

$k_1 = 0.1 \times (y_2 - x_2) = 0.1 \times (2.4210 - 0.2) = 0.2221$,

$k_2 = 0.1 \times (y_2 + k_1 - x_2 - 0.1) = 0.1 \times (2.4210 + 0.2221 - 0.2 - 0.1) =$

0.23431,

$y(0.3) = y_3 \approx y_2 + \frac{1}{2}(k_1 + k_2) = 2.421 + \frac{1}{2}(0.2221 + 0.23421) \approx 2.6492.$

To find $y(0.4) = y_4$:

$k_1 = 0.1 \times (y_3 - x_3) = 0.1 \times (2.6492 - 0.3) = 0.23492,$

$k_2 = 0.1 \times (y_3 + k_1 - x_3 - 0.1) = 0.1 \times (2.6492 + 0.23492 - 0.3 - 0.1) =$

$0.248412,$

$y(0.4) = y_4 \approx y_3 + \frac{1}{2}(k_1 + k_2) = 2.6492 + \frac{1}{2}(0.23492 + 0.248412) \approx 2.8909.$

Answer: $y(0) = 2$, $y(0.1) \approx 2.2050$, $y(0.2) \approx 2.4210$, $y(0.3) \approx 2.6492$, $y(0.4) \approx 2.8909$.

Example:

Find the solution of the initial value problem $\frac{dy}{dx} = 1 + y^2$, $y(0) = 0$ in the interval [0.0.4], by using a fourth order Runge-Kutta method with $h = 0.2$.

Solution: A fourth order Runge-Kutta method is

$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_2 + k_4)$

where

$k_1 = h\, f(x_i, y_i),$

$k_2 = h\, f\left(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}\right),$

$k_3 = h\, f\left(x_i + \frac{h}{2}, y_i + \frac{k_2}{2}\right),$

$k_4 = h\, f(x_i + h, y_i + k_3).$

For the given problem, we have $f(x,y) = 1 + y^2$, $x_0 = 0$, $y_0 = 0$, and $h = 0.2$.

Therefore, the method becomes $y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_2 + k_4)$ where

$k_1 = 0.2 \times (1 + y_i^2)$, $k_2 = 0.2 \times \left(1 + \left(y_i + \frac{k_1}{2}\right)^2\right)$, $k_3 = 0.2 \times \left(1 + \left(y_i + \frac{k_2}{2}\right)^2\right)$,

and $k_4 = 0.2 \times (1 + (y_i + k_3)^2)$.

To find $y(0.2) = y_1$:

$k_1 = 0.2 \times (1 + y_0^2) = 0.2 \times (1 + 0^2) = 0.2$,

$k_2 = 0.2 \times \left(1 + \left(y_0 + \frac{k_1}{2}\right)^2\right) = 0.2 \times \left(1 + \left(0 + \frac{0.2}{2}\right)^2\right) = 0.202$,

$k_3 = 0.2 \times \left(1 + \left(y_0 + \frac{k_2}{2}\right)^2\right) = 0.2 \times \left(1 + \left(0 + \frac{0.202}{2}\right)^2\right) = 0.20204$,

$k_4 = 0.2 \times (1 + (y_0 + k_3)^2) = 0.2 \times (1 + (0 + 0.20204)^2) = 0.20816$,

$y(0.2) = y_1 \approx y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_2 + k_4) = 0 + \frac{1}{6} \times (0.2 + 2 \times 0.202 + 2 \times$

$0.20204 + 0.20816) \approx 0.2027$.

To find $y(0.4) = y_2$:

$k_1 = 0.2 \times (1 + y_1^2) = 0.2 \times (1 + 0.2027^2) \approx 0.2082$,

$k_2 = 0.2 \times \left(1 + \left(y_1 + \frac{k_1}{2}\right)^2\right) = 0.2 \times \left(1 + \left(0.2027 + \frac{0.2082}{2}\right)^2\right) \approx 0.2188$,

$k_3 = 0.2 \times \left(1 + \left(y_1 + \frac{k_2}{2}\right)^2\right) = 0.2 \times \left(1 + \left(0.2027 + \frac{0.2188}{2}\right)^2\right) \approx 0.2195$,

$k_4 = 0.2 \times (1 + (y_1 + k_3)^2) = 0.2 \times (1 + (0.2027 + 0.2195)^2) \approx 0.2356$,

$y(0.4) = y_2 \approx y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_2 + k_4) = 0.2027 + \frac{1}{6} \times (0.2082 + 2 \times$

$0.2188 + 2 \times 0.2195 + 0.2356) \approx 0.4228.$

Answer: $y(0) = 0, y(0.2) \approx 0.2027, y(0.4) \approx 0.4228.$

Remark:

A motivation for Runge-Kutta methods is the Euler method:

$$y_{i+1} = y_i + h\, f(x_i, y_i) = y_i + h\, y_i^{'} = y_i + h \times (slope\ at\ x_i).$$

Runge-Kutta methods are

$$y_{i+1} = y_i + h \times (a\ weighted\ average\ of\ many\ slopes\ at\ points\ \ near\ to\ x_i).$$

These methods are derived by using Taylor series expansions.

Predictor-Corrector methods:

In the previous two methods, we calculated y_{i+1} in terms of known y_i. Such formulae are called explicit formulae. There are many formulae which expresses y_{i+1} in terms of unknown y_{i+1} again. Such formulae are called implicit formulae. However, on many occasions, the order of an implicit formula may be higher than the order of an explicit formula. In such cases, one may like to use an implicit formula instead of convenient explicit formula in order to reduce errors. One standard way to use an implicit formula is to find (predict) an approximate value for y_{i+1} by using an explicit formula, and this value may be substituted in an implicit formula to get an improved (a corrected) value for y_{i+1}. The corrector method may be used again and again by giving an improved value as an input and by getting a further improved value. In this case, we call the explicit formula as a predictor method and the implicit formula as a corrector method. The combination of these two formulae is called a predictor-corrector method. We shall discuss three types of predictor-corrector methods.

Euler's predictor-corrector method:

Consider an initial value problem $y' = f(x,y), y(x_0) = y_0$. Write $x_i = x_0 + ih$, $y_i = y(x_i)$, $f_i = f(x_i, y_i) = f(x_i, y(x_i))$. Then, the Euler's predictor-corrector method is:

$P: y_{i+1} = y_i + h\, f(x_i, y_i)$

$C: y_{i+1} = y_i + \frac{h}{2}[f(x_i, y_i) + f(x_{i+1}, y_{i+1})]$.

Example:

Solve the initial value problem $y' = x^2 + y, y(0) = 1$ on the interval [0, 0.1] by using the Euler's predictor-corrector method with $h = 0.05$ correct to 4 decimals.

Solution: Here $x_0 = 0$, $y_0 = 1$, $h = 0.05$, $x_1 = 0.05$, $x_2 = 0.1$, and $f(x,y) = x^2 + y$. Therefore, the Euler's predictor-corrector method becomes:

$P: y_{i+1} = y_i + 0.05\,(x_i^2 + y_i)$

$C: y_{i+1} = y_i + \frac{0.05}{2}[(x_i^2 + y_i) + (x_{i+1}^2 + y_{i+1})]$.

To find $y(0.05) = y_1$:

We first apply the predictor formula.

$y(0.05) = y_1 \approx y_0 + 0.05(x_0^2 + y_0) = 1 + 0.05 \times (0^2 + 1) = 1.05.$

We now use the corrector formula with this predicted value.

$y(0.05) = y_1 \approx y_0 + 0.025[(x_0^2 + y_0) + (x_1^2 + y_1)] = 1 + 0.025 \times [(0^2 + 1) + (0.05^2 + 1.05)] = 1 + 0.025 \times 2.0525 \approx 1.0513$ (Correct to 4 decimals).

We now again use the corrector formula with this corrected value.

$y(0.05) = y_1 \approx y_0 + 0.025[(x_0^2 + y_0) + (x_1^2 + y_1)] = 1 + 0.025 \times [(0^2 + 1) + (0.05^2 + 1.0513)] = 1 + 0.025 \times 2.0538 \approx 1.0513$ (Correct to 4 decimals).

Since, we do not have a significant improvement, we take $y_1 = y(0.05) \approx 1.0513$.

To find $y(0.1) = y_2$:

We first apply the predictor formula.

$y(0.1) = y_2 \approx y_1 + 0.05(x_1^2 + y_1) = 1.0513 + 0.05 \times (0.05^2 + 1.0513) = 1.0513 + 0.05 \times 1.0538 \approx 1.1040.$

We now use the corrector formula with this predicted value.

$y(0.1) = y_2 \approx y_1 + 0.025[(x_1^2 + y_1) + (x_2^2 + y_2)] = 1.0513 + 0.025 \times [(0.05^2 + 1.0513) + (0.1^2 + 1.1040)] = 1.0513 + 0.025 \times (1.0538 + 1.1140) \approx 1.1055$ (Correct to 4 decimals).

We now again use the corrector formula with this corrected value.

$y(0.1) = y_2 \approx y_1 + 0.025[(x_1^2 + y_1) + (x_2^2 + y_2)] = 1.0513 + 0.025 \times$ $[(0.05^2 + 1.0513) + (0.1^2 + 1.1055)] \approx 1.1055$ (Correct to 4 decimals).

Since, we do not have a significant improvement, we take $y_2 = y(0.1) \approx 1.1055$.

Answer: $y(0) = 1, y(0.05) \approx 1.0513, y(0.1) \approx 1.1055$.

Milne's method:

Consider an initial value problem $y' = f(x,y), y(x_0) = y_0$. Write $x_i = x_0 + ih$, $y_i = y(x_i)$, $f_i = f(x_i, y_i) = f(x_i, y(x_i))$. Then, the Milne's predictor-corrector method is:

$P: y_{i+1} = y_{i-3} + \frac{4h}{3}[2f_{i-2} - f_{i-1} + 2f_i]$

$C: y_{i+1} = y_{i-1} + \frac{h}{3}[f_{i-1} + 4f_i + f_{i+1}]$.

Note that f_{i+1} in C involves y_{i+1}. Note further that we need four initial values to apply this method. If the initial value is given only at x_0, evaluate the values of y at x_1, x_2, x_3, by using Euler's method (or Runge-Kutta methods) before applying this method.

Example:

Find $y(2.0)$ by using Milne's predictor-corrector method for the problem: $\frac{dy}{dx} = \frac{1}{2}(x + y)$. Assume that $y(0) = 2$, $y(0.5) = 2.636$, $y(1.0) = 3.595$, and $y(1.5) = 4.968$.

Solution: The Milne's predictor-corrector method is:

$P: y_{i+1} = y_{i-3} + \frac{4h}{3}[2f_{i-2} - f_{i-1} + 2f_i]$

$C: y_{i+1} = y_{i-1} + \frac{h}{3}[f_{i-1} + 4f_i + f_{i+1}].$

Given: $f(x,y) = \frac{x+y}{2}$, $x_0 = 0$, $h = 0.5$, $x_1 = 0.5$, $x_2 = 1.0$, $x_3 = 1.5$, $y_0 = 2$,

$y_1 = 2.636$, $y_2 = 3.595$, $y_3 = 4.968$.

The predictor formula gives:

$y(2.0) = y_4 \approx y_0 + \frac{4\times0.5}{3} \times \left[2\left(\frac{x_1+y_1}{2}\right) - \left(\frac{x_2+y_2}{2}\right) + 2\left(\frac{x_3+y_3}{2}\right)\right] = 2 + \frac{2}{3} \times$

$\left[2 \times \left(\frac{0.5+2.636}{2}\right) - \left(\frac{1.0+3.595}{2}\right) + 2 \times \left(\frac{1.5+4.968}{2}\right)\right] \approx 6.8710.$

Let us use this predicted value in the corrector formula.

$y(2.0) = y_4 \approx y_2 + \frac{0.5}{3} \times \left[\left(\frac{x_2+y_2}{2}\right) + 4\left(\frac{x_3+y_3}{2}\right) + \left(\frac{x_4+y_4}{2}\right)\right] = 3.595 + \frac{0.5}{3} \times$

$\left[\left(\frac{1+3.595}{2}\right) + 4 \times \left(\frac{1.5+4.968}{2}\right) + \left(\frac{2+6.8710}{2}\right)\right] \approx 6.8732$ (Correct to 4 decimals).

Let us use this corrected value again in the corrector formula.

$y(2.0) = y_4 \approx y_2 + \frac{0.5}{3} \times \left[\left(\frac{x_2+y_2}{2}\right) + 4\left(\frac{x_3+y_3}{2}\right) + \left(\frac{x_4+y_4}{2}\right)\right] = 3.595 + \frac{0.5}{3} \times$

$\left[\left(\frac{1+3.595}{2}\right) + 4 \times \left(\frac{1.5+4.968}{2}\right) + \left(\frac{2+6.8732}{2}\right)\right] \approx 6.8734$ (Correct to 4 decimals).

Since we do not have a significant improvement, we take $y(2.0) \approx 6.8734$.

Answer: $y(2.0) \approx 6.8734$.

Adams-Moulton method:

Consider an initial value problem $y' = f(x,y), y(x_0) = y_0$. Write $x_i = x_0 + ih$, $y_i = y(x_i)$, $f_i = f(x_i, y_i) = f(x_i, y(x_i))$. Then, the Adams-Moulton predictor-corrector method is:

$$P: y_{i+1} = y_i + \frac{h}{24}[55f_i - 59f_{i-1} + 37f_{i-2} - 9f_{i-3}]$$

$$C: y_{i+1} = y_i + \frac{h}{24}[9f_{i+1} + 19f_i - 5f_{i-1} + f_{i-2}].$$

Note that f_{i+1} in C involves y_{i+1}. Note further that we need four initial values to apply this method. If the initial value is given only at x_0, evaluate y_1, y_2, y_3 by using Euler's method (or, Runge-Kutta methods) before applying this method.

Example:

Find the solution of the initial value problem $y' = \frac{1}{2}(x+y), y(0) = 2$, by using the Adams-Moulton predictor-corrector method with $h = 0.5$ in the interval [0, 2].

Solution: Given: $f(x,y) = \frac{1}{2}(x+y)$, $x_0 = 0$, $y_0 = 2$, and $h = 0.5$.

Let us first use the Euler's formula: $y_{i+1} = y_i + h\,f(x_i, y_i)$; to guess the values of y_1, y_2, y_3.

$$y(0.5) = y_1 \approx y_0 + 0.5\left(\frac{x_0+y_0}{2}\right) = 2 + 0.5 \times \left(\frac{0+2}{2}\right) = 2.5,$$

$$y(1.0) = y_2 \approx y_1 + 0.5\left(\frac{x_1+y_1}{2}\right) = 2.5 + 0.5 \times \left(\frac{0.5+2.5}{2}\right) = 3.25,$$

$$y(1.5) = y_3 \approx y_2 + 0.5\left(\frac{x_2+y_2}{2}\right) = 3.25 + 0.5 \times \left(\frac{1+3.25}{2}\right) = 4.3125.$$

So, we take $y_0 = 2$, $y_1 = 2.5$, $y_2 = 3.25$, and $y_3 = 4.3125$ for our further calculations.

Let us now use the Adams-Moulton predictor formula.

$$y(2) = y_4 \approx y_3 + \frac{0.5}{24} \times \left[55\left(\frac{x_3+y_3}{2}\right) - 59\left(\frac{x_2+y_2}{2}\right) + 39\left(\frac{x_1+y_1}{2}\right) - 9\left(\frac{x_0+y_0}{2}\right)\right] =$$

$$4.3125 + \frac{1}{48} \times \left[55 \times \left(\frac{1.5+4.3125}{2}\right) - 59 \times \left(\frac{1+3.25}{2}\right) + 37 \times \left(\frac{0.5+2.5}{2}\right) - 9 \times \left(\frac{0+2}{2}\right)\right] = 4.3125 + \frac{1}{48} \times [159.8438 - 125.375 + 55.5 - 9] \approx 5.994$$ (Correct to 4 decimals).

Let us now use this predicted value in the Adams-Moulton corrector formula.

$$y(2) = y_4 \approx y_3 + \frac{0.5}{24} \times \left[9\left(\frac{x_4+y_4}{2}\right) + 19\left(\frac{x_3+y_3}{2}\right) - 5\left(\frac{x_2+y_2}{2}\right) + \left(\frac{x_1+y_1}{2}\right)\right] =$$

$$4.3125 + \frac{1}{48} \times \left[9 \times \left(\frac{2+5.9994}{2}\right) + 19 \times \left(\frac{1.5+4.3125}{2}\right) - 5 \times \left(\frac{1+3.25}{2}\right) + \left(\frac{0.5+2.5}{2}\right)\right] =$$

$$4.3125 + \frac{1}{48} \times [35.9973 + 55.2188 - 10.625 + 0.75] \approx 6.0071$$ (Correct to 4 decimals).

Let us use this corrected value again in the Adams-Moulton corrector formula.

$$y(2) = y_4 \approx y_3 + \frac{0.5}{24} \times \left[9\left(\frac{x_4+y_4}{2}\right) + 19\left(\frac{x_3+y_3}{2}\right) - 5\left(\frac{x_2+y_2}{2}\right) + \left(\frac{x_1+y_1}{2}\right)\right] =$$

$$4.3125 + \frac{1}{48} \times \left[9 \times \left(\frac{2+6.0071}{2}\right) + 19 \times \left(\frac{1.5+4.3125}{2}\right) - 5 \times \left(\frac{1+3.25}{2}\right) + \left(\frac{0.5+2.5}{2}\right)\right] =$$

$$4.3125 + \frac{1}{48} \times [36.0320 + 55.2188 - 10.625 + 0.75] \approx 6.0078$$ (Correct to 4 decimals).

Since there is no significant improvement, let us take $y(2.0) \approx 6.0078$.

Answer: $y(0) = 2$, $y(0.5) \approx 2.5$, $y(1.0) \approx 3.25$, $y(1.5) \approx 4.3125$, $y(2) \approx$ 6.0078.

The Taylor series method:

Consider an initial value problem: $y' = f(x,y), y(x_0) = y_0$. Then

$y'' = f' = f_x + f_y\, y' = f_x + f_y f$, where $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$.

$y''' = f'' = \frac{d[f_x + f_y f]}{dx} = f_{xx} + f_{xy} f + f_y f_x + [f_{xy} + f_{yy} f + f_y^2]y' = f_{xx} + 2f f_{xy} +$ $f^2 f_{yy} + f_x f_y + f f_y^2$

with $y' = f, f_{xx} = \frac{\partial^2 f}{\partial x^2}$, etc. $y'''' = \cdots$ .... Substitute them in the Taylor series

$$y(x) = y(x_0) + \frac{y'(x_0)}{1!}(x - x_0) + \frac{y''(x_0)}{2!}(x - x_0)^2 + \cdots \ldots \ldots$$

to get a solution at any x in an interval in which the series converges.

Example:

From the Taylor series for $y(x)$, find $y(0.1)$ correct to 4 decimal places, if $y(x)$ satisfies $y' = x - y^2$ and $y(0) = 1$.

Solution:

$y' = x - y^2$

$y'' = 1 - 2yy'$

$y''' = -2yy'' - 2y'^2$

$y^{(iv)} = -2yy''' - 6y'y''$

$y^{(v)} = -2yy^{(iv)} - 8y'y''' - 6y''^2$

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Therefore,

$y(0) = 1,$

$y'(0) = 0 - y(0)^2 = 0 - 1^2 = -1,$

$y''(0) = 1 - 2y(0)y'(0) = 1 - 2 \times 1 \times (-1) = 3,$

$y'''(0) = -2y(0)y''(0) - 2y'(0)^2 = -2 \times 1 \times 3 - 2 \times (-1)^2 = -8,$

$y^{(iv)}(0) = -2y(0)y'''(0) - 6y'(0)y''(0) = -2 \times 1 \times (-8) - 6 \times (-1) \times 3$=34,

$y^{(v)}(0) = -2y(0)y^{(iv)}(0) - 8y'(0)y'''(0) - 6y''(0)^2 = -2 \times 1 \times 34 - 8 \times$

$(-1) \times (-8) - 6 \times 3^2 = -186,$

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Therefore, the Taylor series

$y(x) = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \cdots$ ....

becomes

$y(x) = 1 - x + \frac{3}{2}x^2 - \frac{4}{3}x^3 - \frac{17}{12}x^4 - \frac{31}{20}x^5 + \cdots$ ....

In particular, we have

$y(x) \approx 1 - x + \frac{3}{2}x^2 - \frac{4}{3}x^3 - \frac{17}{12}x^4 - \frac{31}{20}x^5$,

and hence

$y(0.1) \approx 1 - 0.1 + \frac{3}{2} \times 0.1^2 - \frac{4}{3} \times 0.1^3 - \frac{17}{12} \times 0.1^4 - \frac{31}{20} \times 0.1^5 \approx 0.9138$

(Correct to 4 decimals).

Thus, we have $y(0.1) \approx 0.9138$.

Remark:

$y(x_0 + h) \approx y(x_0) + \frac{y'(x_0)}{1!}((x_0 + h) - x_0)$. That is, $y_1 \approx y_0 + h\, f(x_0, y_0)$. This may be modified as $y_{i+1} \approx y_i + h\, f(x_i, y_i)$. Thus, Taylor series leads to Euler's iteration formula. This approach can also be used to derive solutions at the node points. This is illustrated in the next example.

Example:

Solve $y' = x + y, y(0) = 0$ on the interval [0, 0.6] with step length $h = 0.2$ by using a Taylor series method.

Solution: Here $f(x, y) = x + y, x_0 = 0 = y_0$, and $h = 0.2$.

(1) $y' = x + y; y'' = 1 + y' = 1 + x + y; y''' = y'' = 1 + y' = 1 + x + y; \ldots\ldots\ldots$

Since $y(0) = 0$, we have $y'(0) = 0 + y(0) = 0 + 0 = 0$, $y''(0) = 1 + 0 + y(0) = 1$, $y'''(0) = 1 + 0 + y(0) = 1$, ………… Therefore, we have

$y(0.2) \approx y(0) + \frac{y'(0)}{1!}(0.2 - 0)^1 + \frac{y''(0)}{2!}(0.2 - 0)^2 + \frac{y'''(0)}{3!}(0.2 - 0)^3 = \frac{1}{2} \times 0.2^2 + \frac{1}{6} \times 0.2^3 = 0.02 + 0.0001 = 0.0201.$

Let us use (1) again. $y(0.2) = 0.0201, y^{'}(0.2) = 0.2 + y(0.2) = 0.2 + 0.0201 = 0.2201$, $y^{''}(0.2) = 1 + 0.2 + y(0.2) = 1 + 0.2 + 0.0201 = 1.2201$, $y^{'''}(0.2) = 1 + 0.2 + y(0.2) = 1.2201$. Therefore, we have

$y(0.4) \approx y(0.2) + \frac{y^{'}(0.2)}{1!}(0.4 - 0.2)^1 + \frac{y^{''}(0.2)}{2!}(0.4 - 0.2)^2 + \frac{y^{'''}(0.2)}{3!}(0.4 - 0.2)^3 = 0.0201 + 0.2201 \times 0.2 + \frac{1.2201}{2} \times 0.04 + \frac{1.2201}{6} \times 0.008 \approx 0.0201 + 0.0440 + 0.0244 + 0.0016 = 0.0901$.

Let us use (1) again. $y(0.4) = 0.0901, y^{'}(0.4) = 0.4 + y(0.4) = 0.4 + 0.0901 = 0.4901$, $y^{''}(0.4) = 1 + 0.4 + y(0.4) = 1 + 0.4 + 0.0901 = 1.4901$, $y^{'''}(0.4) = 1 + 0.4 + y(0.4) = 1.4901$. Therefore, we have

$y(0.6) \approx y(0.4) + \frac{y^{'}(0.4)}{1!}(0.6 - 0.4)^1 + \frac{y^{''}(0.4)}{2!}(0.6 - 0.4)^2 + \frac{y^{'''}(0.4)}{3!}(0.6 - 0.4)^3 = 0.0901 + 0.4901 \times 0.2 + \frac{1.4901}{2} \times 0.04 + \frac{1.4901}{6} \times 0.008 \approx 0.0901 + 0.0980 + 0.0298 + 0.0020 = 0.2199$.

Answer: $y(0) = 0, y(0.2) \approx 0.0201, y(0.4) \approx 0.0901, y(0.6) \approx 0.2199$.

Picard's method of successive approximations:

This method comes under the heading "approximation".

Consider an initial value problem: $y^{'} = f(x, y), y(x_0) = y_0$. This problem is equivalent to the problem: $y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt$. This means that a solution of one problem becomes a solution of the other problem. We solve the problem of integral equation by a fixed point iteration method.

Write $y_0(x) = y_0$ for all x. Define

$y_1(x) = y_0 + \int_{x_0}^{x} f\big(t, y_0(t)\big)\, dt,$

$y_2(x) = y_0 + \int_{x_0}^{x} f\big(t, y_1(t)\big)\, dt,$

$y_3(x) = y_0 + \int_{x_0}^{x} f\big(t, y_2(t)\big)\, dt,$

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$y_n(x) = y_0 + \int_{x_0}^{x} f\big(t, y_{n-1}(t)\big)\, dt.$

In general, $y_n(x)$ converges to a solution of the integral equation and hence a solution of the given initial value problem as n tends to infinity. Thus, if n is sufficiently large, then $y_n(x)$ is a good approximation to the solution of the given initial value problem.

Example:

Solve $y' = x + y, y(0) = 1$ by using the Picard's method.

Solution: Here $f(x, y) = x + y$, $x_0 = 0$, $y_0 = 1$. Therefore, the Picard's iteration formula

$y_n(x) = y_0 + \int_{x_0}^{x} f\big(t, y_{n-1}(t)\big)\, dt$ becomes $y_n(x) = 1 + \int_{0}^{x} [t + y_{n-1}(t)]\, dt.$

We take $y_0(x) = y_0 = 1$ for all x. Then

$y_1(x) = 1 + \int_0^x (t+1)\, dt = 1 + x + \frac{x^2}{2!}$,

$y_2(x) = 1 + \int_0^x \left(t + 1 + t + \frac{t^2}{2!}\right) dt = 1 + x + x^2 + \frac{x^3}{3!}$,

$y_3(x) = 1 + \int_0^x \left(t + 1 + t + t^2 + \frac{t^3}{3!}\right) dt = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{4!}$,

$y_4(x) = 1 + \int_0^x \left(t + 1 + t + t^2 + \frac{t^3}{3} + \frac{t^4}{4!}\right) dt = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{3\times4} + \frac{x^5}{5!}$,

$y_5(x) = 1 + \int_0^x \left(t + 1 + t + t^2 + \frac{t^3}{3} + \frac{t^4}{3\times4} + \frac{t^5}{5!}\right) dt = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{3\times4} + \frac{x^5}{3\times4\times5} + \frac{x^6}{6!}$.

In general, we have

$y_n(x) = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{3\times4} + \frac{x^5}{3\times4\times5} + \cdots \ldots + \frac{x^n}{3\times4\times5\times\ldots\times n} + \frac{x^{n+1}}{n!}$

$= 1 + x + 2\left\{\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!}\right\} + \frac{x^{n+1}}{(n+1)!}$

$= 2\left[1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\right] - 1 - x + \frac{x^{n+1}}{(n+1)!}$

$\to 2\left[1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \ldots\right] - 1 - x + 0 = 2e^x - 1 - x$, as $n \to \infty$.

Thus, $y(x) = 2e^x - 1 - x$ is the “exact” solution of the given initial value problem.

Example:

Find an approximate solution to the problem $y' = x + y^2,\ y(0) = 1$, by using the Picard’s method.

Solution: Here $f(x,y) = x + y^2$, $x_0 = 0$, $y_0 = 1$. The Picard's formula becomes

$y_n(x) = 1 + \int_0^x [t + (y_{n-1}(t))^2]\, dt$. Take $y_0(x) = y_0 = 1$ for all x. Then

$$y_1(x) = 1 + \int_0^x [t + 1^2]\, dt = 1 + x + \frac{x^2}{2},$$

$$y_2(x) = 1 + \int_0^x \left[t + \left(1 + t + \frac{t^2}{2}\right)^2\right] dt = 1 + x + \frac{3}{2}x^2 + \frac{2}{3}x^3 + \frac{1}{4}x^4 - \frac{1}{20}x^5 .$$

Thus, $y = 1 + x + \frac{3}{2}x^2 + \frac{2}{3}x^3 + \frac{1}{4}x^4 - \frac{1}{20}x^5$ is an "analytic" approximate solution to the given problem.

Higher Order Equations:

We have discussed so far first order initial value problems. Let us now explain how to extend our techniques to solve systems of first order equations, and hence to solve higher order equations.

Consider an initial value problem:

$$\frac{dy}{dx} = f(x,y,z),$$

$$\frac{dz}{dx} = g(x,y,z)$$

subject to $y(x_0) = y_0$, $z(x_0) = z_0$.

Here z and y depend on a single independent variable x. The Euler method can now be extended to solve this problem in the following form:

$$y_{i+1} = y_i + h\, f(x_i, y_i, z_i)$$

$z_{i+1} = z_i + h\ g(x_i, y_i, z_i)$

where $y_i = y(x_i), z_i = z(x_i)$ and $x_i = x_0 + ih$.

A second order initial value problem $y'' = F(x, y, y'), y(x_0) = y_0, y'(x_0) = y_0'$ can be written as

$\frac{dy}{dx} = x$,

$\frac{dz}{dx} = F(x, y, z)$,

$y(x_0) = y_0,\ z(x_0) = y_0'$.

We know how to solve this system by Euler's method. The other methods, other than Euler's method, can also be extended in similar ways to solve systems of first order equations and higher order initial value problems.

Final Remarks:

A fundamental idea in solving differential equations with conditions for unique solutions is replacement of derivatives by difference formulae, and then solving difference equations after simplifying the difference equations by using given conditions with differential equations. One method based on this idea is Euler's method. Such methods are called in general as finite difference methods. Finite difference methods are applicable to solve ordinary differential equations as well as partial differential equations. For partial differential equations, partial derivatives should be replaced by the corresponding finite difference formulae, when finite difference methods are applied. There are many sophisticated methods to solve differential equations. However, almost all basic classic ideas have been discussed in this chapter, except conversion of boundary value problems into initial value problems and linearization of problems.

Exercises:

(1) Solve the initial value problem $y' = -2xy^2,\ y(0) = 1$ on the interval [0, 1] with step length $h = 0.2$, by using (a) The Euler method; (b) A fourth order Runge-Kutta method; (c) The Euler predictor-corrector method; (d) A Taylor series method.

(2) Find the exact solution of the initial value problem $y' = -2xy,\ y(0) = 1$, by using the Picard's method on the interval (-1, 1). Verify that the series converges in that interval by using ratio test or root test for convergence of a power series.

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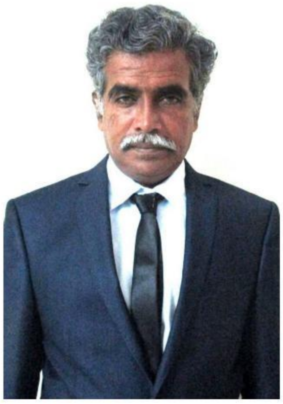
This book is intended for those who completed school studies and who had calculus course and trigonometry course in schools. If a student is particular in developing a software package containing programs to solve numerical problems, he/she should know the fundamental principles for numerical methods. This book also fulfills this need of a student in a way because it provides introductory formal as well as non formal methods. It provides details in worked-out examples so that this book will also be suitable for students who pursue online courses in introductory numerical analysis.



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